PMATH 351 Lecture 1: September 14, 2009

PM351
Real Analysis
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Office Hours
Wed 2:30–3:30
Thursday 3–4

Wed Sept 16
12–1:30
DC 1302
NSERC Scholarships
(due Sept 25)

Definition: Two sets A and B have the same *cardinality* (and write $|A| = |B|$) if there is a bijection between A and B.

Say cardinality of A is $\leq$ cardinality of B (write $|A| \leq |B|$) if there is an injection: $A \to B$.

Cardinality is an equivalence relation:

1. $|A| = |A|$ (reflexive) (identity map)
2. $|A| = |B| \iff |B| = |A|$ (symmetric)
3. $|A| = |B|$ and $|B| = |C| \implies |A| = |C|$

$$A \underset{\substack{1\text{–}1\\\text{onto}}}{\overset{f}{\to}} B \underset{\substack{1\text{–}1\\\text{onto}}}{\overset{g}{\to}} C$$

$$g \circ f \colon A \to C^{1)}$$

Example: Say A has n elements and $|A| = |B|$. Here $f \colon A \to B$ is 1–1, onto.

$\implies$ B has at least n elements, because f is 1–1.

$\implies$ B has at most n elements because f is onto.

Thus B has n elements.

On the other hand, if A and B both have n elements then there exists a bijection: $A \to B$.
Say $A = \{a_1, a_2, \ldots, a_n\}$, $B = \{b_1, b_2, \ldots, b_n\}$.
Define $f(a_j) = b_j$, bijection.
Therefore $|A| = |B|$.

Example: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$
$|\mathbb{N}| \leq |\mathbb{Z}| \leq |\mathbb{Q}| \leq |\mathbb{R}|$
since the embedding maps are injections

$$f \quad \begin{matrix} \mathbb{Z} & 0 & 1 & -1 & 2 & -2 & 3 & -3 & \cdots \\ \mathbb{N} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \end{matrix}$$

$f \colon \mathbb{Z} \to \mathbb{N}$ is a bijection, therefore $|\mathbb{N}| = |\mathbb{Z}|$.

Definition: Say a set A is *countable* if it is either finite or $|A| = |\mathbb{N}|$. Say A is *countably infinite* if countable and infinite.
A is *uncountable* if it is not countable.
e.g., $\mathbb{Z}$ is countable.

1) bijection

Countable sets can be written as $a_1, a_2, a_3, \ldots$

Have $f\colon \mathbb{N} \to A$. Put $a_j = f(j)$.

Conversely, if there is such a list then just define bijection $g\colon A \to \mathbb{N}$ by $g(a_j) = j$.

$\mathbb{Q} = \{\, p/q : p \in \mathbb{Z},\ q \in \mathbb{N},\ (p,q) \text{ coprime} \,\}$, $|\mathbb{Q}| = |\mathbb{N}|$

figure: diagonal winding through $\mathbb{N}^2$

e.g., $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

Problem: $|\mathbb{R}^2| = |\mathbb{R}|$

e.g., Any countable union of countable sets is countable. i.e.,

$$A = \bigcup_{i=1}^{\infty} A_i \qquad |A_i| = |\mathbb{N}|$$

then $|A| = |\mathbb{N}|$

Proof:

$$\begin{aligned} A_i &= \{a_1^{(i)}, a_2^{(i)}, a_3^{(i)}, \ldots\} \\ &= \{a(i,1), a(i,2), \ldots\} \end{aligned}$$

Proposition: If $|A| \le |\mathbb{N}|$ then either A is finite or $|A| = |\mathbb{N}|$.
Corollary: Hence any subset of a countable set is countable.

figure: diagonal winding through $a(i,j)$

PMATH 351 Lecture 2: September 16, 2009

Cardinality
$|A| = |B|$ means there exists a bijection from A to B
$|A| \le |B|$ means there exists an injection from A to B

Countable
either finite or cardinality $= |\mathbb{N}|$
e.g., $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

Proposition: If A is infinite and $|A| \le |\mathbb{N}|$ then $|A| = |\mathbb{N}|$.

Lemma: Every infinite subset B of $\mathbb{N}$ is countably infinite.

Proof: Claim: Every non-empty subset X of $\mathbb{N}$ has a least element.
Why? Pick $n \in X$ and look at $\{\, k \in X : k \le n \,\}$. This is a finite set of positive integers and has a least element k_1. k_1 is the least element of X.

B is non-empty so it has a least element, call it b_1.
$B \setminus \{b_1\}$ is non-empty so it has a least element, call it b_2.
$B \setminus \{b_1, b_2\}$ is non-empty so it has a least element, call it b_3.
Repeat. Produces $b_1 < b_2 < b_3 < \cdots$.
Claim: $B = \{b_n\}_{n=1}^{\infty}$
Why? Take $b \in B$. Look at $\{\, n \in B : n \le b \,\}$[2] $= \{b_1, b_2, \ldots, b_k\}$
$\implies b_k = b$

$$\left.\begin{aligned} \text{Define } f\colon B &\to \mathbb{N} \\ b_n &\mapsto n \end{aligned}\right\} \text{ bijection. Hence } |B| = |\mathbb{N}|.$$

Proof of Proposition: Have an injection $F\colon A \to \mathbb{N}$.
Let $B = F(A) \subseteq \mathbb{N}$.
Note that $F\colon A \to B$ bijection.

[2] say k elements

Hence $|A| = |B|$. Since A is infinite, so is B.
By the lemma $|B| = |\mathbb{N}|$. By transitivity $|A| = |\mathbb{N}|$.

Example: $[0,1) = \{\, x : 0 \le x < 1 \,\}$ is uncountable.

Corollary: $\mathbb{R}$ is uncountable.

Proof: Assume false.

$$\underbrace{[0,1) \underset{\text{injection}}{\subseteq} \mathbb{R} \overset{\text{bijection}}{\to} \mathbb{N}}_{\text{injection}}$$

$$\implies |[0,1)| \le |\mathbb{N}| \implies |[0,1)| = |\mathbb{N}|^{3)}$$

Proof of Example: Suppose $[0,1)$ is countable, say $= \{r_i\}_{i=1}^{\infty}$.

$$r_i = .r_{i1}r_{i2}r_{i3}\cdots \qquad r_{ij} \in \{0,1,\ldots,9\}$$

Let's write a real number not on this list.

$$a = .a_1a_2a_3\cdots$$

$$a_1 = \begin{cases} 8 & \text{if } r_{11} \in \{0,1,\cdots,4\} \\ 1 & \text{if } r_{11} \in \{5,6,\cdots,9\} \end{cases} \quad a_2 = \begin{cases} 8 & \text{if } r_{22} \in \{0,1,\cdots,4\} \\ 1 & \text{if } r_{22} \in \{5,6,\cdots,9\} \end{cases} \quad \cdots \quad a_k = \begin{cases} 8 & \text{if } r_{kk} \in \{0,1,\cdots,4\} \\ 1 & \text{if } r_{kk} \in \{5,6,\cdots,9\} \end{cases}$$

Say $a = r_k$ for some k.
But kth digit of a_k does not agree with kth digit of r_k so $a \ne r_k$.
Thus $\mathbb{R}$ is a different level of infinity.

$$|\mathbb{N}| = \aleph_0 \qquad |\mathbb{R}| = \aleph_1$$

(1) Is $\mathbb{R}$ the "next level" of infinity?

(2) If $A \subseteq \mathbb{R}$, and A is uncountable, is $|A| = |\mathbb{R}|$?

(3) Does there exist a B such that $|\mathbb{N}| < |B| < |\mathbb{R}|$?

Continuum Hypothesis says (2) is yes (and (3) is no).
Answer is independent of set theory axioms.

Given set A, we can define $\mathcal{P}(A) = \{\text{all subsets of } A\}$
e.g., $A = \{0,1\}$, $\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$
If A has n elements then $|\mathcal{P}(A)| = 2^n$

Cantor's Theorem: For any set A, $|A| \le |\mathcal{P}(A)|$ and $|A| \ne |\mathcal{P}(A)|$.
$(|\mathcal{P}(A)| = 1)$

Proof:

$$\text{Injection: } A \to \mathcal{P}(A)$$
$$a \mapsto \{a\}$$

Suppose there is a bijection $g\colon A \to \mathcal{P}(A)$: show this leads to a contradiction.
Let $B = \{\, a \in A : a \notin g(a) \,\}$. $g(a) \in \mathcal{P}(A)$, therefore $g(a)$ is a subset of A.
$B \subseteq A \implies B \in \mathcal{P}(A)$ so there exists $x \in A$ such that $g(x) = B$ because g is onto.
Is $x \in B$?
Try yes: say $x \notin g(x) = B$: contradiction.
So the answer must be no: Means $x \in g(x) = B$: contradiction.
Either way we get contradiction. So there can be no bijection: $A \to \mathcal{P}(A)$.
Therefore $|A| \ne |\mathcal{P}(A)|$.

3) countable

Start with infinite set A

$$|A| < |\mathcal{P}(A)| < |\mathcal{P}(\mathcal{P}(A))| < \cdots$$

Notation: Given set A, write $2^A = \{ f : A \to \{0,1\} \}$
e.g., $|A| = n$, $|2^A| = 2^n = 2^{|A|}$

Theorem: $|\mathcal{P}(A)| = |2^A|$

PMATH 351 Lecture 3: September 18, 2009

$2^A = \{ f : A \to \{0,1\} \}$
If A has n elements then $|\mathcal{P}(A)| = 2^n$ and $|2^A| = 2^n$

Theorem: $|2^A| = |\mathcal{P}(A)|$ for all sets A
Proof: Need to construct bijection $g\colon \mathcal{P}(A) \to 2^A$
Define $g(B) = 1_B$
$B \subseteq \mathcal{P}(A)$ i.e., $B \subseteq A$
where $1_B(x) = \begin{cases} 1 & 1 \text{ if } x \in B \\ 0 & 0 \text{ if } x \notin B \end{cases}$
$1_B \in 2^A$
Check g is 1–1 and onto.
First, if $B \neq C$ then $1_B \neq 1_C$ so $g(B) \neq g(C) \implies g$ is 1–1
Onto: Take $f \in 2^A$
Put $B = \{ x \in A : f(x) = 1 \} \implies f(x) = 1_B(x)$
Therefore $g(B) = f$ where g is a bijection.

Schroeder–Bernstein Theorem
If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.
Proof: Given injections $f\colon A \to B$ and $g\colon B \to A$.

figure:
$D^{\mathrm{C}} = g(f(D)^{\mathrm{C}})$ and
$D = (g(f(E)^{\mathrm{C}}))^{\mathrm{C}}$

$$\text{Define } Q\colon \mathcal{P}(A) \to \mathcal{P}(A)$$
$$E \mapsto (g(f(E)^{\mathrm{C}}))^{\mathrm{C}}$$

Want to find a set D such that $Q(D) = D$.
First, if $E \subseteq F$ then $Q(E) \subseteq Q(F)$ because $f(E) \subseteq f(F) \implies f(E)^{\mathrm{C}} \supseteq f(F)^{\mathrm{C}}$
$\implies g(f(E)^{\mathrm{C}}) \supseteq g(f(F)^{\mathrm{C}}) \implies \underbrace{(g(f(E)^{\mathrm{C}}))^{\mathrm{C}}}_{Q(E)} \subseteq \underbrace{(g(f(F)^{\mathrm{C}}))^{\mathrm{C}}}_{Q(F)}$
Let $\mathcal{D} = \{ E \subseteq A : E \subseteq Q(E) \}$.
Take $D = \bigcup_{E \in \mathcal{D}} E$
If $E \in \mathcal{D}$ then $E \subseteq D$
$\implies Q(E) \subseteq Q(D)$
Also $E \subseteq Q(E) \subseteq Q(D)$ for all $E \in \mathcal{D}$
hence $D = \bigcup_{E \in \mathcal{D}} E \subseteq Q(D)$.
So $D \subseteq Q(D) \implies Q(D) \subseteq Q(Q(D))$
therefore $Q(D) \in \mathcal{D}$.
So $Q(D) \subseteq D$.
Hence $Q(D) = D$
i.e., $D = (g(f(D)^{\mathrm{C}}))^{\mathrm{C}}$ or $D^{\mathrm{C}} = g(f(D)^{\mathrm{C}})$.
Now define $h\colon A \to B$ as follows:

$$h(x) = \begin{cases} f(x) & \text{if } x \in D \\ g^{-1}(x) & \text{for } x \in D^{\mathrm{C}} \text{ and this is well defined because } D^{\mathrm{C}} \subseteq \operatorname{Range} g \end{cases}$$

If $x \in D^{\mathrm{C}}$ then $x \in g(f(D)^{\mathrm{C}})$.
h is 1–1 since both $f|_D$ and $g^{-1}|_{D^{\mathrm{C}}}$ are 1–1 and similarly is onto by construction.
Hence h is a bijection and $|A| = |B|$.

Consequences

1. If $A_1 \subseteq A_2 \subseteq A_3$ and $|A_1| = |A_3|$ then also $|A_2| = |A_3|$.
   Proof: $\underbrace{A_2 \overset{\text{inj}}{\hookrightarrow} A_3}_{\text{embedding}} \implies |A_2| \le |A_3|$

$$\underbrace{A_3 \overset{\text{bij}}{\to} A_1 \overset{\text{inj}}{\hookrightarrow} A_2}_{f}$$

   $f\colon A_3 \to A_2$ is an injection $\implies |A_3| \le |A_2|$
   By S–B, $|A_3| = |A_2|$.

2. $|(0,1)| = |[0,1)| = |\mathbb{R}|$
   $[0,1) \subseteq [0,1) \subseteq \mathbb{R}$.
   So enough to prove $(0,1)$ and $\mathbb{R}$ have same cardinality.
   Let $f(x) = \arctan x$ by $f\colon \mathbb{R} \underset{\text{bij}}{\to} (-\frac{\pi}{2}, \frac{\pi}{2}) \underset{\text{bij}}{\overset{\text{lin}}{\to}} (0,1)$

figure: arctan

figure: alternate definition of f, line between point $(0,1)$ and $r \in \mathbb{R}$, intersects circle with centre $(0,1)$ and radius 1 at $f(r)$

3. $|\mathbb{R}| = |2^{\mathbb{N}}|$, another proof that $\mathbb{R}$ is uncountable.
   Show $|[0,1)| = |2^{\mathbb{N}}|$.
   Given $r \in [0,1)$ write its binary representation

$$r = .a_1a_2a_3\ldots \qquad (\text{where } a_i = 0 \text{ or } 1)$$

   Define $f_r(n) = a_n$. Then $f_r\colon \mathbb{N} \to \{0,1\}$, i.e., $f_r \in 2^{\mathbb{N}}$.

$$\text{Define } \Phi\colon [0,1) \to 2^{\mathbb{N}}$$
$$r \mapsto f_r$$

   Φ is 1–1 because $r_1 \ne r_2$, then there exists n such that nth digits are different, so $f_{r_1}(n) \ne f_{r_2}(n) \implies f_{r_1} \ne f_{r_2}$.

   But Φ is *not* onto because of non-uniqueness of binary representation.

$$\text{Define } \Lambda\colon 2^{\mathbb{N}} \to [0,1)$$
$$f \mapsto .0f(1)0f(2)0f(3)\ldots$$

   Λ is 1–1, since one of the binary representations of a number with two forms ends with a tail of 1s, *and* $\Lambda(f)$ never has a tail of 1s.

   Therefore, by Schroeder–Bernstein, $|2^{\mathbb{N}}| = |\mathbb{R}|$.

PMATH 351 Lecture 4: September 21, 2009

Definition of $\mathbb{R}$:
ordered field, $\supseteq \mathbb{Q}$ and which satisfies the *completeness axiom*: Every increasing sequence that is bounded above converges.

Given sequence (x_n) *bounded above* means exists $r \in \mathbb{R}$ such that $x_n \le r$ for all n.

Converges means there exists $x_0 \in \mathbb{R}$ such that for all $\epsilon > 0$ there exists N such that $|x_n - x_0| < \epsilon$ for all $n \ge N$.

Consequence: *Archimedian Property*: Given any $x \in \mathbb{R}$ there exists $n \in \mathbb{Z}$ such that $x < n$.

Proof: Suppose not. Then there exists a real number r such that $r \ge n$, for all $n \in \mathbb{Z}$. Consider the sequence $\{1$[4]$, 2$[5]$, 3, \ldots\}$. This is a bounded above increasing sequence so by completeness axiom it

[4] x_1
[5] x_2

converges, to say x_0.
Then $|x_n - x_{n-1}|$[6] $\le |x_n - x_0| + |x_0 - x_{n+1}| \le \frac{1}{4} + \frac{1}{4}$ for n large enough. $1 \le \frac{1}{2}$, contradiction.

Example: Use Archimedean property to prove that for real numbers $x < y$,

$$\exists p/q \in \mathbb{Q} \quad \text{such that} \quad x \le p/q < y.$$

Definition: Given $S \subseteq \mathbb{R}$, by an *upper bound* for S we mean $r \in \mathbb{R}$ such that if $x \in S$ then $x \le r$.

If a set has an upper bound we say it is *bounded above*.

Example: $\mathbb{Z}$ has no upper bound.

Example: $S = \left\{ 1 - \frac{1}{n} : n = 1, 2, 3, \dots \right\}$, bounded above by 1 (or 2, or, ...), $1 = \sup(S)$

If a set has an upper bound, then there are infinitely many.

Definition: A *least upper bound* for $S \subseteq \mathbb{R}$ is an upper bound for S, call it B, with the property that whenever $A < B$ then A is not an upper bound for S. Notation: $\text{LUB}(S)$ or $\sup(S)$.

Similarly define greatest lower bound of S, $\text{GLB}(S)$ or $\inf(S)$.

(Exercise) Facts:

1. $\sup(S)$ is unique (if it exists)
2. If B is an upper bound for S and $B \in S$, then $B = \sup S$.
3. If $(x_n)_{n=1}^{\infty}$ is increasing and bounded above, and if $S = \{x_1, x_2, x_3, \dots\}$ then $\sup(S) = \lim_{n\to\infty} x_n$
4. $B = \sup(S)$ iff B is an upper bound for S and $\forall \epsilon > 0 \, \exists x \in S$ such that $x > B - \epsilon$

figure: real line

Completeness Theorem: If $S \subseteq \mathbb{R}$ is non-empty and bounded above then the $\sup(S)$ exists.
"no holes" property of $\mathbb{R}$.

Proof: For this proof use notation z[7] $\ge S$[8] to mean $z \ge x \, \forall x \in S$. Since $S \ne \emptyset$ so $\exists y \in S$. Put $x_0 = y - 1$. Proceed inductively to construct a sequence.
By the Archimedean property and the fact that S is bounded above, there exists $N_0 \in \mathbb{Z}$ such that $x_0 + N_0 \ge S$. In fact, let's make N_0 the least integer that does this. $N_0 \ge 1$ since $x_0 + 0 = y - 1$ and $y \in S$.
Put $x_1 = x_0 + N_0 - 1 \ge x_0$.
By definition of N_0, $x_0 + N_0 - 1$ fails to be $\ge S$. Hence there exists $s_1 \in S$ such that $s_1 > x_0 + N_0 - 1 = x_1$.
Futhermore $x_1 + 1 = x_0 + N_0 \ge S$.
Choose smallest integer N_1 such that $x_1 + N_1/2 \ge S$ ($N_1 = 1$ or 2)
Put $x_2 = x_1 + (N_1 - 1)/2$, fails $\ge S$.
i.e., $\exists s_2 \in S$ with $s_2 > x_2$. Also $x_2 + 1/2 = x_1 + N_1/2 \ge S$.
Inductively define $x_n = x_{n-1} + (N_{n-1} - 1)/n$ where N_{n-1} = least integer such that $x_{n-1} + N_{n-1}/n \ge S$
By construction $\exists s_n \in S$ such that $x_n < s_n$, but $x_n + 1/n \ge S$.

figure: (x_i) on real line

$$\implies N_{n-1} \ge 1 \implies x_{n+1} \ge x_n$$

Produces a sequence (x_n) that is increasing.
If B is an upper bound for S then $x_n \le B$ hence the sequence is bounded above.
By completeness axiom (x_n) converges to say x_0.
Claim: $x_0 = \sup(S)$

1. (x_n) increasing, therefore $x_n \le x_0$, $\forall n$. Say $\exists s \in S$, $s > x_0$. Then $s - x_0 > 1/N$ for some $N \in \mathbb{N}$ $\implies s > 1/N + x_0 \ge 1/N + x_n$, contradiction. Therefore x_0 is an upper bound for S.

[6] $|n - (n+1)| = 1$
[7] $\in \mathbb{R}$
[8] set

2. Show $\forall \epsilon > 0\, \exists x \in S$ such that $x > x_0 - \epsilon$.
   Get x_n such that $x_n > x_0 - \epsilon$ (since $(x_n) \to x_0$).
   Know $\exists s_n \in S$ with $s_n > x_n > x_0 - \epsilon$.
   By our characterization of sup, $x_0 = \sup(S)$.

PMATH 351 Lecture 5: September 23, 2009

Review:
Completeness axiom: Every bounded above, increasing sequence converges.

Completeness Theorem: Every non-empty subset of $\mathbb{R}$ which is bounded above has a LUB or sup.

Definition: A sequence (x_n) is Cauchy if for all $\epsilon > 0$ there exists an N such that for all $n, m \geq N$, $|x_n - x_m| < \epsilon$.

exercise: Cauchy sequences are bounded.
Convergent sequences are Cauchy.

Theorem: (Completeness Property)
Every Cauchy sequence in $\mathbb{R}$ converges.
Say $\mathbb{R}$ is *complete*.

Limit Inferior and Limit Superior:
(x_n) bounded sequence.

Consider the sets $\{x_n, x_{n+1}, \ldots\}$: bounded sets

(because entire sequence is bounded)

Let $A_n = \inf\{x_n, x_{n+1}, \ldots\}$ (exists by completeness)

(then) $A_n \leq A_{n+1} \implies (A_n)_{n=1}^{\infty}$ increasing sequence.

(and) (A_n) is bounded above (UB for original sequence).
By completeness theorem, this sequence converges to

$$\lim_{n\to\infty} A_n = \sup_n A_n,$$

since increasing.

Notation: $\liminf(x_n) \stackrel{\text{def}}{=} \lim_{n\to\infty} A_n = \sup A_n$

[also written as: $\underline{\lim}(x_n)$]

[Reason for terminology lim inf:]

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} (\inf\{x_n, x_{n+1}, \ldots\})$$
$$= \lim_{n\to\infty} \left(\inf_{j\geq n} x_j\right)$$
$$\limsup(x_n)^{9)} \stackrel{\text{def}}{=} \lim_{n\to\infty} (\sup\{x_n, x_{n+1}, \ldots\})$$
$$= \lim_{n\to\infty} \left(\sup_{j\geq n} x_j\right) = \inf_n \left(\sup_{j\geq n} x_j\right)$$
$$\limsup(x_n) \geq \liminf(x_n).$$

Always these exist for bounded sequence.

Example: $x_{2n} = 1 + \frac{1}{2n}$, $x_{2n+1} = \frac{-1}{2n+1}$

figure: x_i on real line

$$\left.\begin{aligned} A_1 &= x_1 \\ A_2 &= x_3 \\ A_3 &= x_3 \\ A_4 &= x_5 \\ A_5 &= x_5 \end{aligned}\right\} \lim A_n = 0 \implies \liminf(x_n) = 0$$

9) $\overline{\lim}(x_n)$

Similarly, $\limsup(x_n) = 1$.

Theorem: $L = \limsup(x_n)$ if and only if $\forall \epsilon > 0$, $x_n < L + \epsilon$, for all but finitely many n, and $x_n > L - \epsilon$ for infnitely many n.

$L = \liminf(x_n)$ if and only if $\forall \epsilon > 0$, $x_n > L - \epsilon$, for all but finitely many n, and $x_n < L + \epsilon$ infinitely often.

Problem:
Theorem: A bounded sequence (x_n) converges if and only if $\liminf x_n = \limsup x_n$, and in this case the common value is $\lim x_n$.

Proof: ($\Longrightarrow$) Say $\lim x_n = L$. This means for all $\epsilon > 0$, there exists N such that

$$|x_n - L| < \epsilon, \qquad \forall n \geq N.$$

i.e., $L - \epsilon < x_n < L + \epsilon$, $\forall n \geq N$.
By our characterization, $L = \limsup(x_n) = \liminf(x_n)$.

($\Longleftarrow$) Suppose $\limsup x_n = \liminf x_n = L$.
We'll see that $L = \lim x_n$.
For $\epsilon > 0$, want to find N such that $|x_n - L| < \epsilon$, $\forall n \geq N$.

Since $L = \limsup x_n$, $\exists N_1$ such that $x_n < L + \epsilon$, $\forall n \geq N_1$.

Similarly, since $L = \liminf x_n$, $\exists N_2$ such that $x_n > L - \epsilon$, $\forall n \geq N_2$.

Take $N = \max(N_1, N_2)$.
Then $\forall n \geq N$, $L - \epsilon < x_n < L + \epsilon$, $\forall n \geq N$.
$\Longrightarrow$ $L = \lim x_n$.

Proposition: Every bounded sequence (x_n) has a subsequence which converges to $\limsup(x_n)$ and (another) subsequence converging to $\liminf(x_n)$.

Proof: Let $L = \limsup x_n$. Know for all k, $x_n < L + 1/k$, $\forall n \geq N_k$, and $x_n > L - 1/k$, infinitely often.

Construct our subsequence: Pick $n_1 > N_1$ such that $x_{n_1} > L - 1/1$. Since $n_1 > N_1$, we have $x_{n_1} < L + 1/1$.

Pick $n_2 > \max(n_1, N_2)$, such that $x_{n_2} > L - 1/2$, and $x_{n_2} < L + 1/2$.

Repeat: Pick $n_k > n_{k-1}$ such that $L + 1/k > x_{n_k} > L - 1/k$.

Consider the sequence $(x_{n_k})_{k=1}^{\infty}$. By construction it converges to L.

Bolzano–Weierstrass Theorem (Corollary): Every bounded sequence has a convergent subsequence.

PMATH 351 Lecture 6: September 25, 2009

Metric Spaces

Definition: A *metric space* is a set X with a metric (or distance function) d with $d\colon X \times X \to [0, \infty)$ satisfying

1. $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$ $\forall x, y \in X$
3. $d(x, y) \leq d(x, z) + d(z, y)$ $\forall x, y, z \in X$, triangle inequality

Examples:

1. $\mathbb{R}$, $d(x, y) = |x - y|$
2. $\mathbb{R}^n$, $d(x, y) = d_2((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2} = \|x - y\|$, *Euclidean metric*

3. $\mathbb{R}^2$, $d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$, $d_1((1,0),(0,1)) = 2$
4. $\mathbb{R}^2$, $d_\infty(x,y) = \max(|x_1 - y_1|, |x_2 - y_2|)$
   triangle inequality:

$$\begin{aligned} |x_1 - y_1| &\le |x_1 - z_1| + |z_1 - y_1| \\ &\le d_\infty(x,z) + d_\infty(z,y) \end{aligned}$$

   Similarly, $|x_2 - y_2| \le d_\infty(x,z) + d_\infty(z,y)$
   $\implies d_\infty(x,y) \le d_\infty(x,z) + d_\infty(z,y)$
   Think about what $\{ x : d_{\_}(x,0) < 1 \}$ looks like.

figure: ∞-norm square, 2-norm circle, 1-norm diamond

5. X any set, d = discrete metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{else} \end{cases}$$

6. $X = \{ x = (x_1, \ldots, x_n) : x_i = 0, 1 \}$
   - 2 element set
     $d(x,y) = \#$ indices i where $x_i \ne y_i$
   - exercise, e.g., $d((0,1,0),(1,1,0)) = 1$
7. $X = \{\text{bounded sequence } (x_n)\} = l^\infty$
   vector space
   $d_\infty(x,y) = \sup_n |x_n - y_n|$
   Example: $x = (x_n) = (1 - 1/n)$, $y = (y_n)$, $y_n = 1/n$
   $d_\infty(x,y) = \sup_n |(1 - 1/n) - 1/n| = 1$
   $c_0 = \{(x_n) \text{ which converge to } 0\} \subseteq l^\infty$
8. $l^2 = \left\{ (x_n)_{n=1}^\infty : \sum |x_n|^2 < \infty \right\}$

$$d(x,y) = \left(\sum_{i=1}^\infty (x_i - y_i)^2 \right)^{1/2} \qquad \langle x, y \rangle = \sum x_i y_i$$

   Define l^p, $1 \le p \le \infty$

$$l^p = \left\{ (x_n) : \sum |x_n|^p < \infty \right\}$$

   Problem: $l^1 \subsetneq l^p \subsetneq c_0 \subsetneq l^\infty$, $1 < p < \infty$
9. X = inner product space

$$d(x,y) = \sqrt{\langle x - y, x - y \rangle}$$

Topology: (X,d) metric space
Ball (centred at x_0 with radius r) in $(\mathbb{R}^2, d_2) = \left\{ x \in \mathbb{R}^2 : d(x, x_0) < r \right\}$
Definition: Given metric space (X,d) we let

$$B(x_0, r) = \{ x \in X : d(x, x_0) < r \}, \qquad r > 0$$

ball centred at x_0, radius r

Example:

1. In $\mathbb{R}$, $|\cdot|$, $\quad B(x_0, r) = (x_0 - r, x_0 + r)$
2. In $\mathbb{R}^2$, d_1, $\quad$ balls are diamonds
3. X, discrete metric, $\quad B(x_0, r) = \{x_0\}$ for $r \le 1$, $B(x_0, r) = X$ for $r > 1$

Definition: Let $U \subseteq X$. Say $x_0 \in U$ is an *interior point of* U if $\exists r > 0$ such that $B(x_0, r) \subseteq U$. Write $\operatorname{int} U$ for set of interior points of U. Say U is *open* if every point of U is an interior point of U.

Example:

1. $\mathbb{R}$ figure: real line $[0, 1)$

$$\begin{aligned} U &= [0,1) \\ \operatorname{int} U &= (0,1) \end{aligned}$$

   Which nonempty intervals are open sets? Open intervals (a, b)

2. $\emptyset$ is always open in any metric space
   X is always open

3. $\mathbb{R}^2$ open in all d_1, d_2, d_∞ figure: open strip in $\mathbb{R}^2$
   Problem: Show that the same open sets are produced by d_1, d_2 or d_∞.

4. X, discrete metric
   $U \subseteq X$, $\operatorname{int} U = U$, since if $x_0 \in U$ then $B(x_0, 1) = \{x_0\} \subseteq U$.
   Hence every set is open.

Proposition: Balls are open sets.

Proof: Consider the ball $B(x_0, r)$ and let $z \in B(x_0, r)$
Put $\rho = r - d(x_0, z) > 0$
Reqired to prove: $B(z, p) \subseteq B(x_0, r)$
Fix $w \in B(z, p)$
Calculate

$$\begin{aligned} d(w, x_0) &\leq d(w, z) + d(z, x_0) \\ &< \rho + d(z, x_0) \\ &= r - d(x_0, z) + d(z, x_0) = r \end{aligned}$$

$\implies d(w, x_0) < r \implies w \in B(x_0, r)$

Hence $B(z, \rho) \subseteq B(x_0, r)$, so z is an interior point of $B(x_0, r)$, and since z was an arbitrary point of $B(x_0, r)$, this proves $B(x_0, r)$ is open.

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Ball $B(x_0, r) = \{\, x \in X : d(x, x_0) < r \,\}$ $(r > 0,\ x_0 \in X)$
$U \subseteq X$ is *open* if $\forall u \in U\ \exists B(u, r) \subseteq U$ for some $r > 0$

Proposition: Balls are open sets.

Proposition:

1. If U_1, U_2 are open then $U_1 \cap U_2$ is open.
2. If $\{U_i\}_{i \in I}$ are open then $\bigcup_{i \in I}$ is open.

Proof:

1. Let $x \in U_1 \cap U_2$. Since $x \in U_i$ and these are open, $\exists r_i > 0$ such that $B(x, r_i) \subseteq U_i$. Let $r = \min(r_1, r_2) > 0$ and then $B(x, r) \subseteq B(x, r_1) \subseteq B(x, r_2) \subseteq U_1 \cap U_2$
   $U_1 \cap U_2$ is open
2. If $x \in \bigcup_{i \in I} U_i$ then $\exists i_0 \in I$ such that $x \in U_{i_0}$. That set is open so $\exists r$ such that $B(x, r) \subseteq U_{i_0} \subseteq \bigcup_{i \in I} U_i$
   $\implies \bigcup U_i$ is open.

Example: $B(0, \frac{1}{n})$ in $\mathbb{R}^2$. $\bigcap_{i=1}^{\infty} B(0, \frac{1}{n}) = \{0\}$, not open.
This shows an infinite intersection of open sets need not be open.

Proposition: U is open iff U is a union of balls.

Proof: ($\Longleftarrow$) Any union of balls is a union of open sets, therefore is open.
($\Longrightarrow$) Since U is open, $\forall x \in U\ \exists B(x, r_x) \subseteq U$.
Claim $U = \bigcup_{x \in U} B(x, r_x)$
RHS $\subseteq U$ as each $B(x, r_x) \subseteq U$
But each $x \in U$ belongs to $B(x, r_x)$, therefore $U \subseteq$ RHS

Proposition: $\operatorname{int} U = \bigcup_{\substack{V \subseteq U \\ \text{open}}}$: says $\operatorname{int} U$ is the largest open subset of U

Proof: Let $x \in \operatorname{int} U$. By definition $\exists r > 0$ such that $B(x, r) \subseteq U$.
$B(x, r)$ is an open set in U therefore $x \in \bigcup_{\substack{V \subseteq U \\ V \text{ open}}} V \longrightarrow \operatorname{int} U \subseteq \bigcup_{\substack{V \subseteq U \\ V \text{ open}}} V$
Pick $x \in \bigcup_{\substack{V \subseteq U \\ V \text{ open}}} V$. Then $x \in V$ some $V \subseteq U$, open.
So $\exists B(x, r) \subseteq V \subseteq U \implies x \in \operatorname{int} U \implies \bigcup_{\substack{V \subseteq U \\ V \text{ open}}} V \subseteq \operatorname{int} V$

$\operatorname{int}(A \cup B) \overset{?}{\neq} \operatorname{int} A \cup \operatorname{int} B$
No:

1. $\underbrace{(-1, 0]}_{A} \cup \underbrace{[0, 1)}_{B}$
   $\operatorname{int}(A \cup B) = (-1, 1)$
   $\operatorname{int} A = (-1, 0)$, $\operatorname{int} B = (0, 1)$
2. $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$
   $\operatorname{int} A = \emptyset = \operatorname{int} B$
   $\operatorname{int}(A \cup B) = \operatorname{int} \mathbb{R} = \mathbb{R}$

Definition: $A \subseteq X$ is *closed* if $A^{\mathrm{C}} = X \setminus A$ is open
Example:

1. $\mathbb{R}$: which intervals are closed sets?

$$[a, b], [a, \infty], (-\infty, a], (-\infty, \infty)$$

$[a, b)$ is not closed because $(-\infty, a) \cup [b, -\infty)$ is not open as b is not an interior point.

2. X, $\emptyset$ are both open and closed
3. $\mathbb{Q} \subseteq \mathbb{R}$ is neither open nor closed
4. (X, d), $\{x_0\}$ is closed
   Proof: Let $z \notin \{x_0\}$, i.e., $z \neq x_0$
   Consider $B(z, d(z, x_0))$. Verify that $x_n \notin B(z, d(z, x_0))$
   That's true since $B(z, d(z, x_0)) = \{\, y : d(y, z) < d(z, x_0) \,\}$ and $y = x_0$ does not have that property.
   Thus $B(z, d(z, x_0)) \subseteq \{x_0\}^{\mathrm{C}}$. Therefore $\{x_0\}$ is closed.

   figure: line between x_0 and z
5. $\{\, x : d(x, x_0) = r_0 \,\}$ is closed
6. Discrete space: Every set is closed (and open)
7. $\mathbb{Z}$, $|\cdot|$, $\quad B(n, r)$[10] $= \{n\}$
   Every set is open and closed.

   figure: $n-1$, n, $n+1$ on real line

Proposition:

1. Any intersection of closed sets is closed.

10) $r \leq 1$

2. A finite union of closed sets is closed.

Proof:

1. Let $U = \bigcap U_i$, U_i closed

$$U^C = \left(\bigcap U_i\right)^C = \underbrace{\bigcup \underbrace{U_i^C}_{\text{open}}}_{\text{open}} \qquad \text{therefore } U \text{ is closed}$$

Definition: A point $x \in X$ is an *accumulation point*[11] *of* $U \subseteq X$ if $\forall r > 0$, $B(x,r) \cap (U \setminus \{x\}) \neq \emptyset$
(i.e., every ball about x contains a point of U other than x)
Equivalently: every open set V containing x satisfies

$$V \cap (U \setminus \{x\}) \neq \emptyset.$$

Equivalently, $\forall r > 0$, $B(x,r) \cap U$ is infinte.

Take $B(x,r)$: Find $u_1 \in B(x,r) \cap (U \setminus \{x\})$.
Consider $B(x, d(x,u_1)) \ni u_2$, where $u_2 \in U \setminus \{x\}$
($u_2 \neq u_1$, since $u_1 \notin B(x, d(x,u_1))$)
Repeat to find a countably infinite set $\{u_i\} \subseteq U$, with $u_i \in B(x,r)$.

figure: radii around point x with u_1, u_2, u_3 increasingly closer to x

Example:

1. $U = [0,1)$ in $\mathbb{R}$
   1 is an accumulation point of U [but 1 is not in U.]
   Everything in U is an accumulation point of U. Nothing else.

   figure: $[0,1)$ real line

2. $U = [0,1) \cup \{2\}$ in $\mathbb{R}$.
   2 is *not* an accumulation point: called *isolated points*.

   figure: $[0,1) \cup \{2\}$ real line

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Accumulation point: $x \in X$ is an accumulation point of $U \subseteq X$ if $\forall r > 0$, $B(x,r) \cap (U \setminus \{x\}) \neq \emptyset$.

Example:

1. $U = [0,1) \cup \{2\}$ in $\mathbb{R}$
   Accumulation points of $U = [0,1]$

   figure: U on real line

2. $\mathbb{Q}$ in $\mathbb{R}$: All points of $\mathbb{R}$ are accumulation points.

3. $U = B(x_0, 1)$ in $\mathbb{R}^2$ with any of these metrics d_1, d_2, d_∞.
   Take $y \in \mathbb{R}^2$ with $d(x_0, y) = 1$
   These points are accumulation points in all 3 cases.
   Now let $U = B(x_0, 1)$ in X.
   Take $y \in X$ with $d(x_0, y) = 1$.
   Is y an accumulation point of U?
   Not if X is the discrete metric space.
   Take $B(y, 1/2) = \{y\}$: Does it intersect U? No.

   figures: y on boundard of $B(x_0, 1)$

4. Any set U in discrete metric space

   - No point is an accumulation point since balls of radius $r \leq 1$ are singletons

   Every point in discrete metric space is isolated.

5. $\mathbb{Z}$: every point is isolated.

[11] (cluster point, limit point)

Theorem: A set U is closed if and only if U contains all its accumulation points.

Corollary:

1. Any finite set is closed
2. In the discrete metric space every set is closed
3. Any set with no accumulation points is closed.

Proof: ($\Longrightarrow$) Assume U is closed. Take $x \notin U$ and show x is not an accumulation point of U.
$x \in U^{\mathrm{C}}$ and this set is open. Hence $\exists r > 0$ such that $B(x,r) \subseteq U^{\mathrm{C}}$. Thus $B(x,r) \cap U = \emptyset$.
Therefore x is not an accumulation point of U.
($\Longleftarrow$) Assume U contains all its accumulation points.
Show U^{C} is open. Take $x \in U^{\mathrm{C}}$. By assumption x is not an accumulation point of U. Hence $\exists r > 0$ such that $B(x,r) \cap U = \emptyset$, i.e., $B(x,r) \subseteq U^{\mathrm{C}}$. $\Longrightarrow$ U^{C} is open $\Longrightarrow$ U is closed.

Notation: $\overline{A}$ = closure of $A = A \cup \{\text{accumulation points of } A\}$

Notes: If A is closed then $\overline{A} = A$
If $\overline{A} = A$ then all accumulation points of A are in A, therefore A is closed.
e.g., $\overline{\mathbb{Q}}$ in $\mathbb{R}$ is $\mathbb{R}$.

Theorem:

1. $\overline{A}$ is a closed set
2. $\overline{A} = \bigcap_{\substack{B \text{ closed} \\ B \supseteq A}} B$

Proof:

1. Show that $\overline{A}^{\mathrm{C}}$ is open.
   Let $x \in \overline{A}^{\mathrm{C}}$. Then x is not in A and even x is not an accumulation point of A.
   Then $\exists r > 0$ such that $B(x,r) \cap A = \emptyset$.
   Claim: $B(x,r) \cap \overline{A} = \emptyset$. Say $y \in B(x,r) \cap \overline{A}$.
   Then y is an accumulation point of A. Since $B(x,r)$ is an open set containing y, it would have to intersect A. But we know it doesn't.
   This proves the claim.
   $$\Longrightarrow\ B(x,r) \subseteq \overline{A}^{\mathrm{C}} \ \Longrightarrow\ \overline{A}^{\mathrm{C}} \text{ is open } \Longrightarrow\ \overline{A} \text{ is closed}$$
2. exercise

Definition: $A \subseteq X$ is *dense* if $\overline{A} = X$
Definition: X is *separable* if it has a countable dense set
e.g., $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{R}$ is separable
Exercise: Show $\mathbb{R}^n$ is separable for all n

1. X discrete metric space: no proper subset is dense since every set is already closed.
2. If A is closed and dense in X, what is A? (any metric space)
   $$\underbrace{A = \overline{A}}_{\text{closed}} \underbrace{= X}_{\text{dense}}$$

Example: $c_0 = \{\, (x_n)_{n=1}^{\infty} : x_n \to 0 \,\} \subseteq l^{\infty}$ = bounded sequences
$d(x,y) = \sup_n |x_n - y_n|$
$l^1 = \{\, (x_n) : \sum |x_n| < \infty \,\} \subseteq c_0$
Show l^1 is dense in c_0.
Take $x = (x_n) \in c_0$ and consider $B(x,r)$

Pick N such that $|x_n| < r$ for all $n \geq N$ and put $y = (x_1, x_2, \dots, x_N, 0, 0, \dots)$
$y \in l^1$

$$\begin{aligned} d(x,y) &= \sup_n |x_n - y_n| \\ &= \sup_{n>N} |x_n - y_n|^{12)} \\ &= \sup_{n>N} |x_n| \\ &< r \end{aligned}$$

This proves $x \in \overline{l^1}$. Therefore l^1 is dense in c_0.

Definition: Bdy $A = \overline{A} \cap \overline{A^{\mathrm{C}}}$

1. Ball in $\mathbb{R}^2$: our "usual" understanding of boundary
2. Bdy $\mathbb{Q}^{13)} = \mathbb{R}$
3. Bdy A, where $A \subseteq X$ discrete metric space: $\overline{A} = A$, $\overline{A^{\mathrm{C}}} = A^{\mathrm{C}}$
   therefore $\overline{A} \cap \overline{A^{\mathrm{C}}} = A \cap A^{\mathrm{C}} = \emptyset$

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Bounded in $\mathbb{R}^n$:
$A \subseteq \mathbb{R}^n$: say A is bounded if $\exists M$ such that $\|x\| < M \ \forall x \in A$
$\iff A \subseteq B(0, M)$ [figure]

Definition: $A \subseteq X$ is *bounded* if $\exists x_0 \in X$ and M such that $A \subseteq B(x_0, M)$
$\iff \forall x \in X \ \exists M_X$ such that $A \subseteq B(x, M_X)$

$$(B(x_0, M) \subseteq B(x, M + d(x_0, x)))$$

Discrete metric space X:
$X \subseteq B(x_0, 1 + \epsilon)$ for any $\epsilon > 0$
X is bounded

Sequences in metric spaces:
Recall definition of convergence of (x_n) in $\mathbb{R}^N$
$\exists x_0 \in \mathbb{R}^N$
$\forall \epsilon > 0 \ \exists M$ such that $\forall n \geq M$
$\|x_n - x_0\|^{14)} < \epsilon$

Definition: Say (x_n) in X *converges* if $\exists x_0 \in X$ such that $\forall \epsilon > 0$
$\exists N$ with $d(x_n, x_0) < \epsilon \ \forall n \geq N$
i.e., $x_n \in B(x_0, \epsilon) \ \forall n \geq N$
Equivalently, the sequence of real numbers $(d(x_n, x_0))_{n=1}^{\infty}$ converges to 0 in $\mathbb{R}$.

Proposition: $(x_n) \to x_0$ if and only if $\forall$ open set U containing x_0, $\exists N$ such that $x_n \in U \ \forall n \geq N$.

Proof: ($\Longrightarrow$) Let U be an open set containing x_0
$\exists \epsilon > 0$ such that $B(x_0, \epsilon) \subseteq U$ (because U is open)
Since $x_n \to x_0$ $\exists N$ such that $x_n \in B(x_0, \epsilon)^{15)} \ \forall n \geq N$

Thus $x_n \in U \ \forall n \geq N$

($\Longleftarrow$) $B(x_0, \epsilon)$ is an open set containing x_0.

---
12) since $x_n = y_n$ for all $n \leq N$
13) $\subseteq \mathbb{R}$
14) $= d(x_n, x_0)$
15) $\subseteq U$

Exercise: Limits are unique.
Convergent sequences are bounded, i.e., $\{ x_n : n = 1, 2, \dots \}$ is a bounded set.

Example: What do convergent sequences in discrete metric spaces look like? Must have $x_n = x_0$ $\forall n \geq N$ for some N

Proposition: $x \in \overline{E}$ iff $x = \lim x_n$ where $x_n \in E$

Proof: $x \in \overline{E}$ iff $\forall n\ B(x, 1/n) \cap E \neq \emptyset$
($\Longrightarrow$) If $x \in \overline{E}$ pick $x_n \in B(x, 1/n) \cap E$: Then (x_n) is a sequence in E converging to x.
($\Longleftarrow$) If $x_n \to x$ then $\forall \epsilon > 0$, $B(x, \epsilon)$ contains all x_n[16], for $n \geq N$
$\Longrightarrow\ B(x, \epsilon) \cap E \neq \emptyset$, $\forall \epsilon > 0$
$\Longrightarrow\ x \in \overline{E}$

Cauchy sequence: (x_n) is Cauchy if $\forall \epsilon > 0\ \exists N$ such that $d(x_n, x_m) < \epsilon\ \forall n, m \geq N$

Exercise: Every convergent sequence is Cauchy.
If a Cauchy sequence has a convergent subsequence, then the (original) sequence converges to the limit of the subsequence.

Example: $X = \mathbb{Q}$, $|\cdot|$
Take $x_n \in \mathbb{Q}$, $x_n \to \sqrt{2}$ in $\mathbb{R}$.
(x_n) is a Cauchy sequence in $\mathbb{Q}$.
But it does not converge (in metric space $\mathbb{Q}$).

Definition: We say X is *complete* if every Cauchy sequence in X converges.
e.g., $\mathbb{R}^n$ is complete
$\mathbb{Q}$ is not complete.
Discrete metric space is complete.

Proposition: Any closed subset E of a complete metric space is complete.

Proof: Let (x_n) be a Cauchy sequence in E
It's also a Cauchy sequence in X. Hence $\exists x_0 \in X$ such that $\lim x_n = x_0$.
By previous proposition $x_0 \in \overline{E} = E$ as E is closed.
Therefore (x_n) converges in E.

Compactness:

Definition: An *open cover* $\{G_\alpha\}$ of a set X is a collection of open sets whose union contains X.

By a *subcover* of an open cover, $\{G_\alpha\}$, we mean a subfamily of the G_αs whose union still contains X.

Definition: We say X is *compact* if every open cover of X has a finite subcover.

Example: $\mathbb{R}$: not compact
$\{ (-n, n) : n \in \mathbb{N} \}$: open cover with no finite subcover
X infinite discrete metric space: not compact, the open cover by singletons has no finite subcover

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Definition: $A \subseteq X$ is *compact* if every open cover of A has a finite subcover.

e.g., $\mathbb{R}$ not compact: $\{ (-n, n) : n \in \mathbb{N} \}$ is an open cover with no finite subcover.

e.g., $(0, 1)$ not compact: $\{ (1/n, 1 - 1/n) : n = 2, 3, \dots \}$

e.g., X any metric space
$A = \{a_1, \dots, a_N\}$ any finite set is compact
Proof: Let $\{G_\alpha\}$ be an open cover of A
For each $j = 1, \dots, N$ there exists G_{α_j} from the collection such that $a_j \in G_{\alpha_j}$. Then $G_{\alpha_1}, \dots, G_{\alpha_N}$ are a finite subcover of A.

[16] $\in E$

e.g., X discrete metric space. Then $A \subseteq X$ is compact if and only if A is finite.

- Saw on Friday that infinite sets in discrete metric space are not compact: just take $\{ B(a,1) : a \in A \}$

Characterization of compactness in $\mathbb{R}^n$:
Theorem: For $A \subseteq \mathbb{R}^n$ the following are equivalent:

(1) A is compact

(2) A is closed and bounded[17)]

(3) Every sequence from A has a convergent subsequence with the limit in A[18)]

Heine–Borel Theorem does not hold true in general metric spaces.

Proposition: Compact sets in metric spaces are alwasys closed.
Proof: Let K be a compact set. Want to prove K^{C} is open.
Let $x \in K^{\mathrm{C}}$.
For all $y \in K$ there exists $r_y > 0$ such that

$$B(x, r_y) \cap B(y, r_y) = \emptyset$$

Consider $\{ B(y, r_y) : y \in K \}$: open cover of K
K is compact so there exists a finite subcover,
i.e., there exists $B(y_1, r_{y_1}), \ldots, B(y_N, r_{y_N})$ such that

$$\bigcup_{j=1}^{N} B(y_j, r_{y_j}) \supseteq K.$$

Let $r = \min(r_{y_1}, \ldots, r_{y_N}) > 0$.
Claim $B(x,r) \cap K = \emptyset$.
Say $z \in B(x,r) \cap K$. Then there exists $j \in \{1, \ldots, N\}$ such that $z \in B(y_j, r_{y_j})$. So $z \in B(x,r) \cap B(y_j, r_{y_j})$, but $B(x,r) \subseteq B(x, r_{y_j})$, i.e., $z \in B(x, r_{y_j}) \cap B(y_j, r_{y_j}) = \emptyset$ by construction.
Contradiction. Hence $B(x,r) \subseteq K^{\mathrm{C}} \implies K^{\mathrm{C}}$ is open $\iff K$ is closed.

Proposition: Closed subsets of compact sets are compact.
Proof: Let F be a closed subset of compact set X.
Take an open cover $\{G_\alpha\}$ of F.
Then the collection of sets G_α together with the open set F^{C} is an open cover of X.[19)]
Let $G_{\alpha_1}, \ldots, G_{\alpha_N}, (F^{\mathrm{C}})$[20)] be a finite subcover of X.
Then $G_{\alpha_1}, \ldots, G_{\alpha_N}$ must cover F.
So the open cover $\{G_\alpha\}$ of F has a finite subcover.
Hence F is compact.

Proposition: Compact sets (in metric spaces) are bounded.
Proof: Let K be compact set and let $x_0 \in K$.
Consider all balls $B(x_0, n)$, $n = 1, 2, 3, \ldots$
If $k \in K$ then $d(x_0, k) < n_0$ for some large enough integer n_0
i.e., $k \in B(x_0, n_0)$. Therefore

$$k \in \bigcup_{n=1}^{\infty} B(x_0, n)$$

$$\implies K \subseteq \bigcup_{n=1}^{\infty} B(x_0, n)$$

17) (1) and (2): Heine–Borel
18) (1) and (3): Bolzano–Weierstrass
19) $\bigcup G_\alpha \cup F^{\mathrm{C}} \supseteq F \cup F^{\mathrm{C}} = X$
20) (because X is compact)

Hence $\{ B(x_0, n) : n = 1, 2, \dots \}$ is an open cover of K.
Since K is compact there must be a finite subcover, say $B(x_0, n_1), \dots, B(x_0, n_L)$.
Say $n_L = \max(n_1, \dots, n_L)$
Then $B(x_0, n_L) \supseteq B(x_0, n_j)$ for $j = 1, 2, \dots, L$
$\implies K \subseteq B(x_0, n_L) = \bigcup_1^L B(x_0, n_j)$
Hence K is bounded.

Definition: *ϵ-net*: for $A \subseteq$ metric space X is a finite set $x_1, \dots, x_n \in X$ such that every element of A has distance at most ϵ from at least one x_j.
i.e., for all $a \in A$ there exists $j \in \{1, \dots, n\}$ such that $d(a, x_j) \leq \epsilon$.
If take $\epsilon' > \epsilon$ then $\bigcup_{j=1}^n B(x_j, \epsilon') \supseteq A$.

Definition: Say A is *totally bounded* if for all $\epsilon > 0$ there exists ϵ-net for A.
e.g., X discrete metric space.
There is a 1-net (consisting of one element)
But no $1 - \epsilon$ net if X is infinite.
So if X is infinite it is not totally bounded.

Proposition: Totally bounded $\implies$ bounded.
Proof: Take a 1-net for the totally bounded set A, say $x_1, \dots, x_k$.
$\implies \bigcup_{j=1}^k B(x_j, 3/2) \supseteq A$
Take $B(x_1, \underbrace{\max_{j=1,\dots,k} d(x_1, x_j) + 1 + 3/2}_{r}) \supseteq B(x_j, 3/2)$ for all j.
Then $A \subseteq B(x_1, r)$

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Totally bounded
ϵ-net: for a set $A \subseteq X$ is a finite set $\{x_1, \dots, x_n\} \subseteq X$ such that for all $x \in A$ there exists j such that $d(x_j, a) \leq \epsilon$.

Totally bounded means A has an ϵ-net for all $\epsilon > 0$.

Totally bounded $\implies$ bounded.

Bounded $\not\Longrightarrow$ Totally bounded: as discrete metric space is bounded, but not totally bounded.

Example: $A =$ **Ball in $\mathbb{R}^2$**

figure: circle with ϵ-grid

Take the set of bottom left corner points from the squares of the ϵ-grid that intersect the ball A. Call this finite set $\{x_1, \dots, x_N\}$.

$$\overline{B(x_j, \sqrt{2}\epsilon)} \supseteq \text{square that } x_j \text{ is a corner of}$$

So $\bigcup_{j=1}^N \overline{B(x_j, \sqrt{2}\epsilon)} \supseteq A$
hence $\{x_1, \dots, x_N\}$ are an $\sqrt{2}\epsilon$-net for A. $\to A$ totally bounded.
Same idea works for a ball in $\mathbb{R}^n$.

Fact: If $U \subseteq V$ and V is totally bounded, then U is totally bounded.
Proof: Take same ϵ-net for U as for V.

Proposition: In $\mathbb{R}^n$, bounded $\implies$ totally bounded.
Proof: A bounded set is a subset of a ball, and balls in $\mathbb{R}^n$ are totally bounded.

Proposition: Compact $\implies$ totally bounded
Proof: Let A be compact. Consider $\{ B(x, \epsilon) : x \in A \}$. This is an open cover for A, so there is a finite subcover, say $B(x_1, \epsilon), \dots, B(x_n, \epsilon)$, i.e., $\bigcup_1^n B(x_j, \epsilon) \supseteq A$
$\implies \{x_1, \dots, x_n\}$ are an ϵ-net for A.

Exercise: A bounded $\implies \overline{A}$ bounded.

Proposition: A totally bounded, then $\overline{A}$ is totally bounded.
Proof: Let $\{x_1, \dots, x_n\}$ be an ϵ-net for A.
Given $x \in \overline{A}$, there exists $a \in A$ such that $d(x, a) < \epsilon$.
$\exists j$ such that $d(x_j, a) \leq \epsilon$
Therefore $d(x, x_j) \leq d(x, a) + d(a, x_j) < 2\epsilon$
So $\{x_1, \dots, x_n\}$ are an 2ϵ-net for $\overline{A}$.

Goal is to prove metric spaces are compact if and only if it is complete and totally bounded.

Note: For $A \subseteq \mathbb{R}^n$, A is complete if and only if A is closed
Proof:

1. In any metric space complete implies closed because of the following argument. Let x be an accumulation point of the complete space A. Get $\{a_n\} \subseteq A$ such that $a_n \mapsto x$. Then (a_n) is a Cauchy sequence in the complete space A. By definition of completeness there exists $a \in A$ such that $a_n \to a$. By uniqueness of limits, $x = a \in A$.
   Therefore A is closed.

2. Any closed subset of a complete metric space is complete. In particular, any closed subset of $\mathbb{R}^n$ is complete.

Theorem (Cantor's): If $A_1 \supseteq A_2 \supseteq \cdots$ are non-empty, closed sets in a complete metric space X and

$$\operatorname{diam} A_n = \sup\{\, d(x, y) : x, y \in A_n \,\} \to 0,$$

then $\bigcap_{n=1}^{\infty} A_n$ is exactly one element.

e.g., To see "closed" is necessary, take $A_n = (0, 1/n)$. Here $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

figure: open sets on real line

Proof: Pick $x_n \in A_n$. If $k \geq N$, then $x_k \in A_k \subseteq A_N$. So $\{\, x_k : k \geq N \,\} \subseteq A_N \implies d(x_j, x_k) \leq \operatorname{diam} A_N$ if $j, k \geq N$.
i.e., $\{x_n\}$ is Cauchy and therefore converges[21] to some $x_0 \in X$. Consider the subsequence $(x_n)_{n=N}^{\infty} \subseteq A_N$ and has the same limit x_0. But A_N is closed, therefore $x_0 \in A_N$. This is true for all N, therefore $x_0 \in \bigcap_{N=1}^{\infty} A_N$.
Now suppose $x_0, y_0 \in \bigcap_{n=1}^{\infty} A_n$.
Then $x_0, y_0 \in A_n$ for all n, so $d(x_0, y_0) \leq \operatorname{diam} A_n$[22] for all n.
$\implies d(x_0, y_0) = 0 \implies x_0 = y_0$.

Definition: A collection of sets has the F.I.P. (*finite intersection property*) if every finite intersection is non-empty.
e.g., nested family of sets.

$*$ **Theorem:** The following are equivalent for a metric space X:

(1) X is compact.

(2) Every collection of closed subsets of X with the F.I.P. has non-empty intersection.

(3) Every sequence in X has a convergent subsequence (limit in X)[23]

(4) X is complete and totally bounded.

Corollary: (Heine–Borel): In $\mathbb{R}^n$, compact $\iff$ closed and bounded.

Corollary: compact $\implies$ closed and bounded.
(since complete $\implies$ closed, and totally bounded $\implies$ bounded).

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[21] $\to 0$ as $N \to \infty$
[22] $\to 0$
[23] (1) and (3): Bolzano–Weierstrass Theorem

Theorem: The following are equivalent for a metric space X:

(1) X is compact

(2) Every collection of closed subsets of X with the F.I.P. has non-empty intersection.

(3) Every sequence in X has a convergent subsequence (limit in X)

(4) X is complete and totally bounded

$1 \iff 4$: Analogue of the Heine–Borel
$1 \iff 3$: Bolzano–Weierstrass Theorem

Cantor's Intersection Theorem
If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ are non-empty, closed subseteq of a complete metric space X and

$$\operatorname{diam} A_n \equiv \sup_n \{\, d(x,y) : x, y \in A_n \,\} \to 0$$

then $\bigcap_{n=1}^{\infty} A_n$ is one point.

Proof: ($4 \implies 1$): Suppose X is not compact. Say $\{U_\alpha\}$ is an open cover of X that has no finite subcover.
Notation: $D(x_0, r) = \{\, x \in X : d(x, x_0) \le r \,\}$
Exercise: closed set
X is totally bounded so there is a $\frac{1}{2}$-net for X, say $\{x_1^{(1)}, \dots, x_{n_1}^{(1)}\}$

$$\text{so } \bigcup_{j=1}^{n_1} D\big(x_j^{(1)}, \tfrac{1}{2}\big) = X.$$

Since there are only finitely many closed balls $D(x_j^{(1)}, \frac{1}{2})$, $j = 1, \dots, n$, needed to cover X, at least one of these balls cannot be covered by only finitely many U_α.
Say $D(x_1^{(1)}, \frac{1}{2}) \equiv X_0$: closed set.
Notice $\operatorname{diam} X_0 = 1 = \frac{1}{2^0}$.
$X_0 \subseteq X$ so X_0 is totally bounded.
Let $\{x_1^{(2)}, \dots, x_{n_2}^{(2)}\}$ be a $\frac{1}{4}$-net for X_0.
Hence $\bigcup_{j=1}^{n_2} D(x_j^{(2)}, \frac{1}{4}) \cap X_0 = X_0$.
At least one of the sets $D(x_j^{(2)}, \frac{1}{4}) \cap X_0$ is not covered by only finitely many U_αs,
say $D(x_1^{(2)}, \frac{1}{4}) \cap X_0 \equiv X_1$.
X_1[24] $\subseteq X_0$, $\operatorname{diam} X_1 \le \frac{1}{2} = \frac{1}{2^1}$
Repeat to get closed sets $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$
$\operatorname{diam} X_j \le \frac{1}{2^j}$ *and* each set X_j cannot be covered by only finitely many U_α.
Each X_j is non-empty (else could cover with finitely many U_αs).
By Cantor's intersection theorem,

$$\bigcap_{n=1}^{\infty} X_n = \{x_0\} \qquad \text{(singleton)}$$

Since $\bigcup U_\alpha = X$, there exists α_0 such that $x_0 \in U_{\alpha_0}$.
As U_{α_0} is open there exists $\epsilon > 0$ such that $B(x_0, \epsilon) \subseteq U_{\alpha_0}$.
Take n such that $\frac{1}{2^n} < \epsilon$ and consider X_n, $\operatorname{diam} X_n \le \frac{1}{2^n}$. If $y \in X_n$ then because $x_0 \in X$ we have $d(x_0, y) \le \operatorname{diam} X_n \le \frac{1}{2^n} < \epsilon \implies y \in B(x_0, \epsilon)$.
So $X_n \subseteq B(x_0, \epsilon) \subseteq U_{\alpha_0}$.
Hence X_n is covered by only one set U_{α_0}: contradiction to choice of X_n.
Thus X must be compact.

[24] closed

($1 \implies 2$): Recall the sets $\{U_\alpha\}$ have the FIP if any finite intersection of these sets is non-empty.

Let $\{A_\alpha\}$ be closed subsets of X and suppose $\bigcap_\alpha A_\alpha = \emptyset$. We will prove some finite intersection is empty.

$$A_\alpha^{\mathrm{C}}\text{: open sets}$$
$$\left(\bigcup A_\alpha^{\mathrm{C}}\right)^{\mathrm{C}} = \bigcap A_\alpha = \emptyset$$
$$\implies \bigcup A_\alpha^{\mathrm{C}} = X$$

hence the sets $\{A_\alpha^{\mathrm{C}}\}$ are an open cover of X.
By compactness (1) there exist infinitely many sets

$$A_{\alpha_1}^{\mathrm{C}}, \dots, A_{\alpha_n}^{\mathrm{C}} \text{ such that } \bigcup_{i=1}^{n} A_{\alpha_i}^{\mathrm{C}} = X$$

$$\implies \bigcap_{i=1}^{n} A_{\alpha_i} = \left(\bigcup_{i=1}^{n} A_{\alpha_i}^{\mathrm{C}}\right)^{\mathrm{C}} = \emptyset$$

($2 \implies 3$): Let (x_n) be a sequence in X.
Define $S_n = \{\, x_k : k \geq n \,\}$
$\overline{S_n}$: non-empty, closed, $\overline{S_n} \subseteq \overline{S_{n-1}}$
Exercise: $A \subseteq B \implies \overline{A} \subseteq \overline{B}$
$\bigcap_1^N \overline{S_k} = \overline{S_N}$, hence any finite intersection is non-empty. Therefore $\{S_n\}$ has FIP.
By assumption (2), $\bigcap_{n=1}^\infty \overline{S_n} \neq \emptyset$. Say $x \in \bigcap_1^\infty \overline{S_n} \implies x \in \overline{S_n}$ for all n. So given any $\epsilon > 0$ and any n, there exists $y_n \in S_n$ such that $d(x, y_n) < \epsilon$. Note $y_n = x_k$ for some $k \geq n$.
Start with $n = 1$, $\epsilon = 1$. Get $y_1 \in S_1$ such that $d(x, y_1) < 1$, say $y_1 = x_{k_1}$.
Take $n = k_1 + 1$, $\epsilon = \frac{1}{2}$.
Find $y_n \in S_n$ such that $d(x, y_n) < \frac{1}{2}$
$y_n = x_{k_2}$ with $k_2 \geq n > k_1$
Repeat with $n = k_2 + 1$, $\epsilon = \frac{1}{4}$ and get x_{k_3} such that $d(x_{k_3}, x) < \frac{1}{4}$ and $k_3 > k_2$.
This produces $k_1 < k_2 < \cdots$, and terms x_{k_j} such that $d(x_{k_j}, x) < \frac{1}{2^{j-1}}$.
$\{x_{k_j}\}_{j=1}^\infty$ is a subsequence of $\{x_n\}$, and clearly $x_{k_j} \to x$.
Hence the sequence (x_n) has a convergent subsequence.

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Theorem: The following are equivalent

1. X is compact
3. Every sequence X has a convergent subsequence (limit in X)
4. X is complete and totally bounded

To finish the proof do ($3 \implies 4$)

(i) Prove X is complete.
Let (x_n) be a Cauchy sequence in X.
By assumption (3), (x_n) has a convergent subsequence. A Cauchy sequence with a convergent subsequence converges.
$\implies$ X is complete.

(ii) Prove X is totally bounded.
Assume not. Then for some $\epsilon > 0$ there is no ϵ-net.
Take $x_1 \in X$. Then $\{x_1\}$ is not an ϵ-net.
So there exists $x_2 \in X$ such that $d(x_1, x_2) > \epsilon$.
Consider $\{x_1, x_2\}$: not an ϵ-net.

So there exists $x_3 \in X$ such that $d(x_1, x_2) > \epsilon$ and $d(x_2, x_3) > \epsilon$.
Repeat: Get $\{x_n\}_{n=1}^{\infty}$ such that $d(x_n, x_j) > \epsilon$ for all $j = 1, \ldots, n-1$, i.e., $d(x_i, x_j) > \epsilon$ for all $i \neq j$.
This sequence has no Cauchy subsequence, so no convergent subsequence: contradicting assumption (3).

Example: Cantor Set $\subseteq [0, 1]$.

- compact, empty interior
  perfect $\to$ closed set in which every point is an accumulation point.

Construction: $C_0 = [0, 1]$
$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ C_2 = union of $4 = 2^2$ intervals of length $\frac{1}{9} = \frac{1}{3^2}$ — figures of C_0, C_1, C_2
C_n = union of 2^n closed intervals, each of length 3^{-n} with gap between any two intervals $\geq 3^{-n}$

C_n is closed $\subseteq [0, 1]$, therefore compact.
$C_n \subseteq C_{n-1}$
Cantor set $C = \bigcap_{n=1}^{\infty} C_n$: closed $\subseteq [0, 1]$, therefore compact.
$0, 1 \in C$. $\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \ldots \in C$: C contains all endpoints of Cantor intervals.

Empty interior: Say $I = (a, b) \subseteq C$.
$\implies I \subseteq C_n$ for all n.
Pick n such that $3^{-n} < b - a = |I|$.
But then $I \not\subseteq C_n$ since the longest intervals in C_n are length 3^{-n}.
$\implies$ contradiction

Perfect: Let $x_0 \in C$. Fix $\epsilon > 0$.
Pick n such that $3^{-n} < \epsilon$.
$x_0 \in C_n \implies x_0$ lies in a Cantor interval of step n, of length 3^{-n}.
$a, b \in C$ — x_0 between a and b, in an interval of length 3^{-n}
$d(x_0, a), d(x_0, b) \leq 3^{-n} < \epsilon$
Hence $B(x_0, \epsilon) \cap (C \setminus \{x_0\})$ is non-empty.
Since $B(x_0, \epsilon) \cap C \supseteq \{a, b\}$

Proposition: A non-empty, perfect set E in $\mathbb{R}^k$ is uncountable.
Proof: E must be infinite since it has accumulation points.
Assume $E = \{x_n\}_{n=1}^{\infty}$ (i.e., E is countably infinite)
Put $k_1 = 1$.
Look at $B(x_{k_1}, 1) = B(x_1, 1) \equiv V_1$: open set containing x_1.
Since x_1 is an accumulation point of E_1 there exists $e \in V_1 \setminus \{x_1\}$, $e \in E$
Pick least integer $k_2 > k_1$ such that $x_{k_2} \in V_1 \cap E$, $x_{k_2} \neq x_{k_1}$
Pick V_2 open, contains x_{k_2} and satisfies $\overline{V_2} \subseteq V_1$ and $x_{k_1} \notin \overline{V_2}$. — figure: x_{k_1} in V_1 and x_{k_2} in V_2
(e.g., $V_2 = B(x_{k_2}, r)$ where $r = \frac{1}{2}\min(d(x_{k_1}, x_{k_2}), 1 - d(x_{k_1}, x_{k_2})))$

Consider $V_2 \cap E \setminus \{x_{k_2}\}$: non-empty
Pick minimal k_3 such that $x_{k_3} \in V_2 \cap E \setminus \{x_{k_2}\}$.

By construction $k_3 > k_2$. — $x_{k_2} \notin \overline{V_3}$
Assume we have chosen $x_{k_n} \in E \cap V_{n-1} \setminus \{x_{k_{n-1}}\}$ with $k_n > k_{n-1}$ and minimal; open sets $V_n \ni x_{k_n}$.
$\overline{V_n} \subset V_{n-1}$ and $x_{k_{n-1}} \notin \overline{V_n}$.
As x_{k_n} is an accumulation point of E, we can choose k_{n+1} minimal such that $x_{k_{n+1}} \in V_n \cap E \setminus \{x_{k_n}\}$.
Then $k_{n+1} > k_n$.
Get V_{n+1} open such that $\overline{V_{n+1}} \subset V_n$ and $x_{k_n} \notin \overline{V_{n+1}}$

$$\text{Put } K_n = \overline{V_n} \cap E^{25)}$$
$$\subseteq V_{n-1} \cap E \subseteq \overline{V_{n-1}} \cap E = K_{n-1}$$

so $K_1 \supseteq K_2 \supseteq \cdots$

[25] non-empty, closed

$$K_n \subseteq K_1 \subseteq \overline{B(x_0,1)}^{26)}.$$

Since nested, have FIP. By characterization of compactness (2), $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

Now, $x_1 \notin \overline{V_2}$, therefore $x_1 \notin \bigcap K_n$; $x_2 \notin V_1$, therefore $x_2 \notin \bigcap K_n$. $x_{k_2} \notin \overline{V_3}$, therefore $x_{k_3} \notin \bigcap K_n$. $x_{2+1} \notin V_2, \ldots$; $x_{k_j} \in \overline{V_{j+1}}$, therefore $x_{k_j} \notin \bigcap K_n$.
$\Longrightarrow$ $x_j \notin \bigcap K_n$, for any j, and $K_n \subseteq E$.
Therefore $\bigcap K_n = \emptyset$: contradiction.

PMATH 351 Lecture 14: October 16, 2009

Midterm: Friday October 23 here at 1:30.
Up to end of compactness.

Additional office hours Tuesday 2–3.

Not proof of 1) Schroeder–Bernstein, 2) Perfect set in $\mathbb{R}^k$ are uncountable.

Continuity: $f\colon X \to Y$, X, Y metric spaces

Definition: Say f is *continuous at* $x_0 \in X$, if for all $\epsilon > 0$ there exist $\delta > 0$ such that whenever $d_X(x_0, y) < \delta^{27)}$ then $d_Y(f(x_0, f(y))) < \epsilon^{28)}$.

figure: f takes a point in a ball in X to one in Y

Say f is *continuous* if it is continuous at every point of its domain.

Examples:

1. Constant functions are always continuous.
2. Identity map: $X \to X$. Take $\delta = \epsilon$.
3. Identity map: $(\mathbb{R}, \text{usual metric})^{29)} \to (\mathbb{R}, \text{discrete metric})^{30)}$
   - not continuous
     Take $\epsilon \leq 1$, then $B_Y(\text{Id}(x_0)^{31)}, \epsilon) = \{x_0\}$.
     So to have $\text{Id}(y) = y \in B_Y(x_0, \epsilon)$ means $y = x_0$.
     But for all $\delta > 0$, $B_X(x_0, \delta)$ contains infinitely many points.
     So it contains some $y \neq x_0$. But then $\text{Id}(y) \notin B_Y(\text{Id}(x_0), \epsilon)$.
4. If x_0 is not an accumulation point of X then any f is continuous at x_0.
   Proof: If $\delta > 0$ is small enough as $B(x_0, \delta) = \{x_0\}$, then clearly if $y \in B(x_0, \delta)$ then $f(y) \in B(f(x_0), \epsilon)$ for all $\epsilon > 0$
   Corollary: If $f\colon X \to Y$ where X is the discrete metric space then f is continuous.
5. (X, d) any metric space and $a \in X$.
   Then $f(x) = d(a, x)$ is continuous, where $f\colon X \to \mathbb{R}$.
   Proof:

$$\begin{gathered} f(x) - f(y) = d(a,x) - d(a,y) \\ \leq d(a,y) + d(x,y) - d(a,y) = d(x_0,y) \\ f(y) - f(x) \leq d(x,y) \\ \Longrightarrow |d(a,x)^{32)} - d(a,y)^{33)}| \leq d(x,y) \end{gathered}$$

   So take $\delta = \epsilon$.

26) compact in $\mathbb{R}^k$
27) $y \in B(x_0, \delta)$
28) $f(y) \in B(f(x_0), \epsilon)$
29) X
30) Y
31) x_0
32) $= f(x)$
33) $= f(y)$

Proposition: f is continuous at x if and only if whenever (x_n) is a sequence in X converging to x; then the sequence $(f(x_n))$ converges to $f(x)$.
Proof: ($\Longrightarrow$) Let $x_n \to x$.
Take $\epsilon > 0$. Get δ by continuity so that $d(x,y) < \epsilon \implies d(f(x), f(y)) < \epsilon$.
Get N such that $d(x_n, x) < \delta$ for all $n \geq N$.
Take $n \geq N$, then $d(f(x_n), f(x)) < \epsilon$ by definition of N and δ.
($\Longleftarrow$) Suppose f is not continuous at x. Then there exists $\epsilon > 0$ such that for every $\delta > 0$ there exists $y = y(\delta)$ with $d(x,y) < \delta$ but $d(f(x), f(y)) \geq \epsilon$.

Take $\delta = \frac{1}{n}$ and put $x_n = y(\frac{1}{n})$.
Then $d(x, x_n) < \frac{1}{n}$, so $x_n \to x$.
But $d(f(x), f(x_n)) \geq \epsilon \implies f(x_n) \not\to f(x)$
Contradiction.

Exercise: $f, g \colon X \to \mathbb{R}$ continuous then so are $f \pm g$, fg, f/g if $g(x) \neq 0$.

Alternate way to look at continuity:
f continuous at x_0 if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$f(B(x_0, \delta)) \subseteq B(f(x_0), \epsilon)$$

if and only if $B(x_0, \delta) \subseteq f^{-1}$[34]$(B(f(x_0), \epsilon))$, where $f^{-1}(v) = \{\, x : f(x) \in V \,\}$.
$\implies x_0 \in \operatorname{int} f^{-1}(B(f(x_0), \epsilon))$

Theorem: The following are equivalent: for $f \colon X \to Y$

1. f is continuous
2. for all V open in Y, $f^{-1}(V)$ is open in X.
3. for all F closed in Y, $f^{-1}(F)$ is closed in X.

Proof: ($1 \implies 2$): Let V be open in Y, and suppose $x_0 \in f^{-1}(V)$, i.e., $f(x_0) \in V$.
Hence there exists $\epsilon > 0$ such that $f(B(x_0, \delta)) \subseteq B(f(x_0, \epsilon)) \subseteq V$.
By continuity, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subseteq B(f(x_0), \epsilon) \subseteq V$.

$$\begin{aligned} \implies B(x_0, \delta) \subseteq f^{-1}(V) &\implies x_0 \text{ is an interior point of } f^{-1}(V) \\ &\implies f^{-1}(V) \text{ is open.} \end{aligned}$$

PMATH 351 Lecture 15: October 19, 2009

Continuity
$f \colon X \to Y$ is *continuous* at x if $\forall \epsilon > 0\ \exists \delta > 0$ such that $f(B(x, \delta)) \subseteq B(f(x), \epsilon) \iff B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon))$

Theorem: $f \colon X \to Y$. The following are equivalent:

1. f is continuous
2. $\forall V$ open in Y, $f^{-1}(V)$ is open in X.
3. $\forall F$ closed in Y, $f^{-1}(F)$ is closed in X.

Proof: ($1 \implies 2$): ✓
($2 \implies 1$): For each $x \in X$, check that f is constant at x.
Put $V = B(f(x), \epsilon)$: open in Y
By (2), $f^{-1}(B(f(x), \epsilon))$ is open in X.

$x \in f^{-1}(B(f(x), \epsilon))$ so since the set is open there exists $\delta > 0$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon))$, i.e., f is continuous at $x \in X$.

[34] preimage

$(2 \implies 3)$: Let F be a closed set in Y.
F^{C} is open set in Y. By (2), $f^{-1}(F^{\mathrm{C}})$ is open in X.

$$f^{-1}(F^{\mathrm{C}}) = \left\{ x \in X : f(x) \in F^{\mathrm{C}} \right\} = \{ x : f(x) \notin F \} = \left\{ x : x \notin f^{-1}(F) \right\} = X \setminus f^{-1}(F) = \underbrace{(f^{-1}(F))^{\mathrm{C}}}_{\text{open}}$$

$\implies f^{-1}(F)$ is closed

Corollary: If $f\colon X \to Y$, $g\colon Y \to Z$, continuous then $g \circ f\colon X \to Z$ is continuous.
Proof: Let $V \subseteq Z$ be open. $(g \circ f)^{-1}(V) = \{ x : g(f(x)) \in V \}$
$\iff f(x) \in g^{-1}(C) \iff x \in f^{-1}(\underbrace{g^{-1}(V)}_{\text{open}})$

$\to$ open as f, g are continuous

figure: $X \xrightarrow{f} Y \xrightarrow{g} Z \subseteq V$ and $g^{-1}(v)$ takes V to Y and $f^{-1}(g^{-1}(v))$ takes Y to X

Examples:

1. $f\colon (0,1) \to \mathbb{R}$
   $x \mapsto 1$
2. $f\colon \underset{\text{closed}}{\mathbb{R}} \to \underset{\text{onto open set}}{(-\frac{\pi}{2}, \frac{\pi}{2})}$
   $f(x) = \arctan(x)$
3. $f\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to \underset{\text{onto}}{\mathbb{R}}$
   $f(x) = \tan x$

open does not necessarily go to open

closed does not have to go to closed

bounded $\not\Longrightarrow$ bounded

Theorem: Let $f\colon K \to X$ be continuous and K compact. Then $f(K)$ is compact.
Proof: Let $\{U_\alpha\}$ be an open cover of $f(K)$.
Then $f^{-1}(U_\alpha)$ are open because f is continuous.
If $x \in K$, then $f(x) \in f(K)$ so $f(x) \in U_\alpha$ for some $\alpha \implies x \in f^{-1}(U_\alpha)$. Hence $\{f^{-1}(U_\alpha)\}$ form an open cover of K.
Since K is compact there is a finite subcover, say $f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_n})$.
Then $U_{\alpha_1}, \ldots, U_{\alpha_n}$ are a finite subcover of $f(K)$ because if $f(x) \in f(K)$ for some $x \in K$ then $x \in f^{-1}(U_{\alpha_i})$ (since these cover K), i.e., $f(x) \in U_{\alpha_i}$.
Hence $f(K)$ is compact.

Corollary: (E.V.T.) If K is compact and $f\colon F \to \mathbb{R}$ is continuous then f attains minimum and maximum values.

Proof: $f(K)$ is compact in $\mathbb{R}$, i.e., closed and bounded.
Let $a = \sup f(K)$ and $b = \inf f(K)$ (exist as $f(K)$ is bounded)
$a, b \in f(K)$ since it is closed,
i.e., $\exists x_1, x_2 \in K$ such that $a \in f(x_1)$, $b = f(x_2)$

Corollary: If $f\colon K \to \mathbb{R}$ is continuous, K compact and $f > 0$ on K then $\exists \delta > 0$ such that $f(x) > \delta$ $\forall x \in K$.

Proof: Take $\delta = f(x_1)$ where $f(x_1) =$ minimum value of f on K.

Corollary: If $f\colon X \to Y$ continuous bijection, X compact, then f is a homeomorphism, i.e., f^{-1} is also continuous.

Proof: $(f^{-1})^{-1}(F^{35)}) = f(F)$

$f^{-1}\colon Y \to X \subseteq F \underset{(f^{-1})(F)}{\to} Y$

Let $F \subseteq X$ be closed. But X is compact, therefore F is compact.
Here $f(F)$ is compact and hence closed. Thus $(f^{-1})^{-1}(F)$ is closed, so f^{-1} is continuous.

Example:

$$f\colon [0, 2\pi) \to \text{boundary unit ball in } \mathbb{R}^2$$
$$t \mapsto (\cos t, \sin t)$$

35) closed

- bijection
- continuous

figure: unit circle

But f^{-1} is not continuous
$f^{-1}(1,0)=0$,
but $f^{-1}(\cos(2\pi-\epsilon),\sin(2\pi-\epsilon))=2\pi-\epsilon$.

Uniform Continuity
Definition: f is *uniformly continuous* if $\forall\epsilon>0$, $\exists\delta>0$ such that if $d(x,y)<\delta$, then $d(f(x),f(y))<\epsilon$. [i.e., δ is *independent of* x]

Note: Uniform continuity $\implies$ continuity; but not conversely.

Example:

1. $f(x)=\frac{1}{x}$ on $(0,1)$ is continuous, but not uniformly continuous.
2. $f(x)=x^2$ on $\mathbb{R}$ is continuous, but not uniformly continuous.

Example 1: Prove it is not uniformly continuous.
Take $\epsilon=1$. Suppose $\delta<1$ worked.
Take $x=\frac{\delta}{2}$, $y=\frac{\delta}{4}$. Then $d(x,y)<\delta$.
But $|f(x)-f(y)|=|\frac{2}{\delta}-\frac{4}{\delta}|=\frac{2}{\delta}>1=\epsilon$,
Example 3: $f\colon[a,1]\to\mathbb{R}$ $(a>0)$
$f(x)=\frac{1}{x}$: Is uniformly continuous.

$$|f(x)-f(y)|=\left|\frac{1}{x}-\frac{1}{y}\right|=\left|\frac{y-x}{xy}\right|\le\frac{|y-x|}{a^2}\le\frac{\delta}{a^2}\le\epsilon.$$

Take $\delta=\epsilon a^2$.

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Proposition: Let X be compact and $f\colon X\to Y$ continuous. Then f is uniformly continuous.

Proof: Let $\epsilon>0$. $\forall x\in X$ $\exists\delta_x>0$ such that if $d(x,y)<\delta_x$ then $d(f(x),f(y))<\epsilon$.
Look at $\{\,B(x,\delta_x/2):x\in X\,\}$: open cover of compact set X.
Take a finite subcover, say $B(x_1,\delta_{x_1}/2),\dots,B(x_n,\delta_{x_n}/2)$
Let $\delta=\min(\delta_{x_1}/2,\dots,\delta_{x_n}/2)>0$
Suppose $d(x,y)<\delta$. There is some i such that $x\in B(x_i,\delta_{x_i}/2)\implies d(x,x_i)<\delta_{x_i}/2<\delta_{x_i}$ so by choice of δ_{x_i}, $d(f(x),f(x_i))<\epsilon$.
Calculate $d(y,x_i)\le d(y,x)+d(x,x_i)<\delta_{x_i}/2+\delta_{x_i}/2=\delta_{x_i}$
$\implies d(f(y),f(x_i))<\epsilon$

$$\text{Hence } d(f(x),f(y))\le d(f(x),f(x_i))+d(f(y),f(x_i))<\epsilon+\epsilon=2\epsilon$$

$\implies f$ is uniformly continuous.

Connectedness:
Definition: X is *not connected* if $X=U\cup V$ where U,V are both open and non-empty and $U\cap V=\emptyset$.

Note $U^{\mathrm{C}}=V$ and $V^{\mathrm{C}}=U$, therefore U,V are closed also.

$E\subseteq X$ is *connected* means $E\ne(E\cap U)\cup(E\cap V)$ where U,V open in X, $E\cap U$, $E\cap V$ are disjoint and $E\cap U$, $E\cap V$ are both non-empty.

Example:

1. $E=(0,1)\cup(2,3)$: not connected
2. $\mathbb{Q}=(\mathbb{Q}\cap(-\infty,\sqrt{2}))\cup(\mathbb{Q}\cap(\sqrt{2},\infty))$
3. X: discrete metric space: only[36)] singletons are connected

4. $[a,b]$ in $\mathbb{R}$ is connected.
   Suppose not, say $[a,b] = (U \cap [a,b]) \cup (V \cap [a,b])$, U, V open, $U \cap [a,b]$ and $V \cap [a,b]$ disjoint, $U \cap [a,b]$, $V \cap [a,b]$ non-empty
   Without loss of generality $b \in U \cap [a,b]$. Let $t = \sup([a,b] \cap V)$
   $([a,b] \cap V)^{\mathrm{C}} = (-\infty, a) \cup (b, \infty) \cup U$: open: $[a,b] \cap V$ is closed
   $t \in [a,b] \cap V$ $\quad t \neq b$ since $b \in U \cap [a,b]$ and the two sets are disjoint.
   $t < b$ $\quad$ So because V is open $\exists \delta > 0$ such that $t + \delta \in V$ *and* $t + \delta < b$
   $\implies t + \delta \in V \cap [a,b]$: contradicts definition of t as $\sup V \cap [a,b]$

Proposition: If X is connected and $f\colon X \to Y$ is continuous then $f(X)$ is connected.

Proof: Suppose not, say $f(X) = A \cup B$, A, B open, disjoint and non-empty
$f^{-1}(A)$, $f^{-1}(B)$

- open as f is continuous
- non-empty as A, B are non-empty
- disjoint because A, B are disjoint

$X = f^{-1}(A) \cup f^{-1}(B)$ as $f(X) = A \cup B$: contradicts assumption X is connected

Path Connected
X is *path connected* if $\forall x \neq y \in X$ there exists an interval $[a,b]$ and continuous function $f\colon [a,b] \to X$ such that $f(a) = x$, $f(b) = y$.

Proposition: path connected implies connected
Proof: Say $X = A \cup B$, A, B open, disjoint and non-empty.
Let $x \in A$, $y \in B$. Let $f\colon [a,b] \to X$ be a path from x to y.

figure: path between x and y in set X

$$\begin{array}{c}
f([a,b]) \quad \text{is connected as } f \text{ is continuous and } [a,b] \text{ is connected} \\
\| \\
(f[a,b] \cap A) \cup (f[a,b] \cap B) \\
\begin{array}{cc} \ni & \ni \\ x & y \\ (\text{as } f(a) = x) & (f(b) = y) \end{array}
\end{array}$$

so these sets are non-empty and disjoint because A, B are disjoint
contradiction

Example: of a connected set that is not path connected

$$X = \left\{ \left(x, \sin \frac{1}{x} \right) : x > 0 \right\} \cup \{(0,0)\}$$

figure: graph of X

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Example: $X = \underbrace{\left\{ (x, \sin \tfrac{1}{x}) : x > 0 \right\}}_{\equiv E} \cup \{(0,0)\}$
Show X is connected, but not path connected.
$X = \overline{E}$

graph of $\sin \frac{1}{x}$ for $x > 0$

Proof outline:

1. E path connected $\implies$ E connected $\implies$[37] $\overline{E}$ connected
2. X is not path connected

[36] (non-empty sets?)
[37] exercise

1. E path connected
   Let $(x_1, \sin\frac{1}{x_1}), (x_2, \sin\frac{1}{x_2}) \in E$ $(x_1, x_2 > 0)$

$$\text{Define } f\colon [0,1] \to E$$
$$t \mapsto \Big(\underbrace{tx_1 + (1-t)x_2}_{>0}, \sin\frac{1}{tx_1 + (1-t)x_2)}\Big) \in E$$

   f continuous on $[0,1]$
   $f(1) = (x_1, \sin\frac{1}{x_1})$, $f(0) = (x_2, \sin\frac{1}{x_2}) \implies E$ is path connected

2. X not path connected
   Prove no "path" joining $(0,0)$ to $(\frac{1}{\pi}, 0)$
   Suppose $f\colon [a,b] \to X$ is a path with $f(a) = (0,0)$, $f(b) = (\frac{1}{\pi}, 0)$
   Claim:
$$\left(\frac{1}{\frac{5\pi}{2}}, 1\right), \left(\frac{1}{\frac{9\pi}{2}}, 1\right), \dots, \left(\frac{1}{\frac{\pi}{2} + 2\pi k}, 1\right) \in f[a,b]$$
$k \in \mathbb{N}$

   Note: $f[a,b]$ is connected as f is continuous and $[a,b]$ is connected.

   Suppose without loss of generality $\left(\frac{1}{\frac{5\pi}{2}}, 1\right) \notin f[a,b]$.
   Then
$$f[a,b] = \Bigg(\overbrace{f[a,b] \cap \left\{ (x,y) : x > \frac{1}{\frac{5\pi}{2}} \right\}}^{\ni(\frac{1}{\pi},0)}\Bigg) \cup \Bigg(\overbrace{f[a,b] \cap \left\{ (x,y) : x < \frac{1}{\frac{5\pi}{2}} \right\}}^{\ni(0,0)}\Bigg)$$

   because only $(x,y) \in X$ with $x = \frac{1}{\frac{5\pi}{2}}$ is the point $\left(\frac{1}{\frac{5\pi}{2}}, 1\right) \notin f[a,b]$

   - this contradicts the fact $f[a,b]$ is connected

   Also $f[a,b]$ is compact.
   The sequence $\left\{\left(\frac{1}{\frac{\pi}{2}+2\pi k}, 1\right)\right\}_{k=1}^{\infty}$ is Cauchy and therefore converges as $f[a,b]$ is complete.
   Hence $(0,1) \in f[a,b] \subseteq X$.
   But $(0,1) \notin X$ so contradiction.

Finite Dimensional Normed Vector Spaces over $\mathbb{R}$ (or $\mathbb{C}$)

Norm on a vector space:

1. $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$
2. $\|\alpha v\| = |\alpha|\|v\|$ for all α scalars, $v \in V$
3. $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$ for all $v_1, v_2 \in V$

Norms always give metrics by $d(x,y) = \|x - y\|$

Example: Space of polynomials on $[0,1]$ of degree $\leq n$

1. $\|p\|_\infty = \max_{x\in[0,1]}|p(x)|$
2. $\|p\|_1 = \int_0^1 |p(x)|\,\mathrm{d}x$

Theorem: Suppose V is a finite dimensional normed vector space over $\mathbb{R}$ with basis $\{v_1, \dots, v_n\}$. Then there exists constants $A, B > 0$ such that for all $(a_1, \dots, a_n) \in \mathbb{R}^n$.

$$A\|(a_1, \dots, a_n)\|_{\mathbb{R}^n} \leq \Big\|\sum_{i=1}^n a_i v_i\Big\|_V \leq B\|(a_1, \dots, a_n)\|_{\mathbb{R}^n}$$

Given any $v \in V$ there exists exactly one $(a_1, \dots, a_n)$ such that $v = \sum_1^n a_i v_i$. Theorem says $\|a_1, \dots, a_n\|_{\mathbb{R}^n} \sim \|v\|_V$
Proof:

$$\begin{aligned}
\Big\|\sum_{i=1}^n a_i v_i\Big\|_V &\le \sum_{i=1}^n \|a_i v_i\|_V \\
&= \sum_{i=1}^n |a_i| \|v_i\|_V \\
&\le^{38)} \left(\sum_{i=1}^n |a_i|^2\right)^{1/2} \left(\sum_{i=1}^n \|v_i\|^2\right)^{1/2} \\
&= \|(a_1, \dots, a_n)\|_{\mathbb{R}^n} B \qquad \text{where } B = \left(\sum_{i=1}^n \|v_i\|^2\right)^{1/2}
\end{aligned}$$

Define $F \colon \mathbb{R}^n \to \mathbb{R}$ by

$$F(a_1, \dots, a_n) = \Big\|\sum_{i=1}^n a_i v_i\Big\|$$

Check F is continuous:

$$\begin{aligned}
F(\boldsymbol{x}) - F(\boldsymbol{y}) &= \Big\|\sum_{i=1}^n x_i v_i\Big\| - \Big\|\sum_{i=1}^n y_i v_i\Big\| \\
&\le \Big\|\sum x_i v_i - \sum y_i v_i\Big\| + \Big\|\sum y_i v_i\Big\| - \Big\|\sum y_i v_i\Big\| \\
&= \Big\|\sum (x_i - y_i) v_i\Big\|
\end{aligned}$$

Similarly $F(y) - F(x) \le \|\sum (x_i - y_i) v_i\|$

$$\begin{aligned}
\implies |F(x) - F(y)| &\le \Big\|\sum (x_i - y_i) v_i\Big\| \\
&\le \sum |x_i - y_i| \|v_i\| \\
&\le \left(\sum |x_i - y_i|^2\right)^{1/2} \underbrace{\left(\sum \|v_i\|^2\right)^{1/2}}_{B} \\
&= B \|\boldsymbol{x} - \boldsymbol{y}\|_{\mathbb{R}^n} \\
&= B d(x, y)
\end{aligned}$$

$\implies F$ is continuous
Restrict F to $S = \{ x \in \mathbb{R}^n : \|\boldsymbol{x}\| = 1 \}$

$$F(x) = 0 \iff x = 0$$

In particular, if $x \in S$ then $F(x) > 0$.
S is compact. By Extreme Value Theorem there exists $\delta > 0$ such that $F(x) \ge \delta$ for all $x \in S$
Take any $a = (a_1, \dots, a_n) \in \mathbb{R}^n \setminus \{0\}$
$\frac{a}{\|a\|_{\mathbb{R}^n}} \in S$.
$F\big(\frac{a}{\|a\|}\big) \ge \delta$.

$$\begin{aligned}
\Big\|\sum a_i v_i\Big\|_V &= \Big\| \|a\|_{\mathbb{R}^n} \sum \frac{a_i}{\|a_i\|_{\mathbb{R}^n}} v_i \Big\|_V \\
&= \|a\|_{\mathbb{R}^n} \Big\|\sum \frac{a_i}{\|a\|} v_i\Big\|_V \\
&= \|a\|_{\mathbb{R}^n} F\left(\frac{a}{\|a\|}\right) \\
&\ge \|a\|_{\mathbb{R}^n} \delta
\end{aligned}$$

[38] Cauchy–Schwartz

Take $A = \delta$.

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Theorem: If V an n dimensional normed vector space over $\mathbb{R}$ with basis $\{v_1, \dots, v_n\}$ then there exists A, B such that

$$A\|(a_1, \dots, a_n)\|_{\mathbb{R}^n} \leq \left\|\sum_{i=1}^{n} a_i v_i\right\|_V \leq B\|(a_1, \dots, a_n)\|_{\mathbb{R}^n}$$

If $T \colon \mathbb{R}^n \to V$
$T(a_1, \dots, a_n) = \sum_{i=1}^n a_i v_i$[39]
then $A\|\boldsymbol{a}\| \leq \|T(\boldsymbol{a})\|_V \leq B\|\boldsymbol{a}\|_{\mathbb{R}^n}$

$$A\|a-b\|_{\mathbb{R}^n} \leq \|T(\boldsymbol{a}-\boldsymbol{b})\|_V = \|T(a) - T(b)\|_V \leq B\|a-b\|_{\mathbb{R}^n}$$

$$Ad(a,b) \leq d(T(a), T(b)) \leq Bd(\boldsymbol{a}, \boldsymbol{b})$$

See that $x_k \to x_0$ if and only if $T(x_k) \to T(x_0)$
So topologies are the same.
Boundedness if the same.
Both T and T^{-1} are continuous so V is homeomorphic to $\mathbb{R}^n$

Corollary: Subset of a finite dimensional vector space is compact if and only if it is closed and bounded.
Corollary: Any finite dimensional subspace of a normed vector space is complete.
Proof: Let V be normed vector space and W finite dimensional subspace. Let $T \colon \mathbb{R}^n \to W$ be a homeomorphism as above.
Let $\{w_k\}$ be a Cauchy sequence in W.
Then $\{x_k = T^{-1}(w_k)\}$ is a Cauchy sequence in $\mathbb{R}^n$.
So there exists x_0 such that $x_k \to x_0$. But then $T(x_k) \to T(x_0) \in W$.
Hence W is complete.

Function Spaces
Convergence: $f_n, f \colon X \to Y$. X, Y metric spaces.
Say $f_n \to f$ *pointwise* if for all $\epsilon > 0$ and for all $x \in X$ there exists N such that $d_Y(f_n(x), f(x)) < \epsilon$ for all $n \geq N$.
i.e., $(f_n(x)) \to f(x)$ for each $x \in X$ (as sequences in Y)
Say $f_n \to f$ *uniformly* if for all $\epsilon > 0$ there exists N such that $d_Y(f_n(x), f(x)) < \epsilon$ for all $x \in X$ and for all $n \geq N$.

Example: $f_n \colon [0,1] \to \mathbb{R}$
$f_n(x) = x^n$

graph of $f_n(x)$ for n increasing

$$f_n \to f = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

- convergence is pointwise, but not uniform

Note: each f_n is continuous, but f is not

Theorem: If f_n are continuous, and $f_n \to f$ uniformly, then f is continuous.
Proof: Fix $\epsilon > 0$ and $x \in X$. Need to find δ such that $d_X(x,y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$
Pick N such that $d(f_n(y), f(y)) < \epsilon/3$ for all $n \geq N$ and for all $y \in X$.
Get $\delta > 0$ such that $d(x,y) < \delta \implies d(f_N(x), f_N(y)) < \epsilon/3$.
Check if this δ works.
Suppose $d(x,y) < \delta$ and look at $d(f(x), f(y)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y), f(y)) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$

Corollary: If g_k are continuous and $\sum g_k$ converges uniformly to g, then g is continuous.
Proof: $S_N = \sum_1^N g_k$ is continuous and $S_N \to g$ uniformly by assumption.

[39] linear, bijection

Definition: A sequence $f_n\colon X \to Y$ is *uniformly Cauchy* if for all $\epsilon > 0$ there exists N such that $d(f_n(x), f_m(x)) < \epsilon$ for all $n, m \geq N$ and for all $x \in X$.

Theorem: Suppose X, Y are metric spaces and Y is complete. Then the sequence $f_n\colon X \to Y$ is uniformly Cauchy if and only if (f_n) is uniformly convergent.
Proof: ($\Longleftarrow$) Say $f_n \to f$ uniformly and pick N such that $d(f_n(x), f(x)) < \epsilon/2$ for all $n \geq N$ and for all $x \in X$.
Then

$$\begin{aligned} d(f_n(x), f_m(x)) &\leq d(f_n(x), f(x)) + d(f(x), f_m(x)) \\ &< \epsilon/2 + \epsilon/2 \qquad \text{if } n, m \geq N \end{aligned}$$

($\Longrightarrow$) Since (f_n) is uniformly Cauchy, then $(f_n(x))$ is Cauchy in Y for each $x \in X$.
Y is complete so there exists $a_x \in Y$ such that $f_n(x) \to a_x$.
Put $f(x) = a_x$ so $f\colon X \to Y$.
Show $f_n \to f$ uniformly.
For $\epsilon > 0$, get N such that $d(f_n(x), f_m(x)) < \epsilon/2$ for all $x \in X$, $\forall n, m \geq N$ (by uniform Cauchy)
Let $n \geq N$ and look at $d(f_n(x), f(x))$ (for arbitrary x)
Get $m > N$ such that $d(f_m(x), f(x)) < \epsilon/2$[40)]
So

$$\begin{aligned} d(f_n(x), f(x)) &\leq d(f_n(x), f_m(x)) + d(f_m(x), f(x)) \\ &< \epsilon/2 + \epsilon/2 = \epsilon \qquad (\text{as } n, m \geq N) \end{aligned}$$

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Corollary: Weierstrass M-test

Let $f_n\colon X \to \mathbb{R}$. If there exists a sequence M_k such that $|f_k(x)| \leq M_k$ for all $x \in X$ and for all k and if $\sum_1^\infty M_k$ converges, then $\sum_{k=1}^\infty f_k$ converges uniformly.

Example:

$$f_k(x) = \frac{\sin kx}{k^2} \qquad |f_k(x)| \leq \frac{1}{k^2} \qquad 0 \leq \sum \frac{1}{k^2} < \infty$$

$\Longrightarrow$ $\sum \frac{\sin kx}{k^2}$ is a continuous function.
Proof: Let $S_N(x) = \sum_1^N f_k(x)$. Show $\{S_N\}$ converges uniformly. It's enough to prove $\{S_N\}$ is uniformly Cauchy.

$$|S_N - S_M(x)| = \left|\sum_{N+1}^{M} f_k(x)\right| \leq \sum_{k=N+1}^{M} \left|f_k(x)\right| \leq \sum_{k=N+1}^{M} M_k \to 0 \text{ as } M > N \to \infty$$

$\Longrightarrow$ $\{S_N\}$ is uniformly Cauchy.

Dini's Theorem: Suppose K is compact and $f_n\colon K \to \mathbb{R}$ converges pointwise to f. If f_n, f are continuous and $f_{n+1}(x) \leq f_n(x)$ for all n, for all $x \in K$, then $f_n \to f$ uniformly.
Proof: Let $g_n = f_n - f$
g_n is continuous
$g_n \to 0$ pointwise
$g_n(x) \geq g_{n+1}(x)$
$g_n \geq 0$ since $f(x) \leq f_n(x)$ as $f_n(x)$ decreases

Prove $g_n \to 0$ uniformly to conclude $f_n \to f$ uniformly.
Let $\epsilon > 0$. Find N such that $|g_n(x)| < \epsilon$ for all $n \geq N$ and for all $x \in K$,
$\Longleftrightarrow$ $0 \leq g_n(x) \leq \epsilon$ for all $n \geq N$ and for all $x \in K$.
Since $g_n \to 0$ pointwise, for all $t \in K$ there exists N_t such that $0 \leq g_n(t) < \frac{\epsilon}{2}$ for all $n \geq N_t$.
In particular, $g_{N_t}(t) < \frac{\epsilon}{2}$.

[40)] depends on x temporarily looking at

Because g_{N_t} is continuous at t so there exists $\delta_t > 0$ such that if $d(t,x) < \delta_t$ then $|g_{N_t}(t) - g_{N_t}(x)| < \frac{\epsilon}{2}$. The balls $B(t, \delta_t)$, $t \in K$ are an open cover of the compact set K. Take a finite subcover say $B(t_1, \delta_{t_1}), \dots, B(t_L, \delta_{t_L})$.

If $x \in K$ there exists i such that $x \in B(t_i, \delta_{t_i})$

$$\begin{aligned} &\implies d(x,t_i) < \delta_{t_i} \implies |g_{N_{t_i}}(t_i) - g_{N_{t_i}}(x)| < \frac{\epsilon}{2} \\ &\implies |g_{N_{t_i}}(x)| \le |g_{N_{t_i}}(x) - g_{N_{t_i}}(t_i)| + |g_{N_{t_i}}(t_i)| \\ &\qquad < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Take $N = \max(N_{t_1}, \dots, N_{t_L})$.

Let $n \ge N$ and $x \in K$. Get t_i as before.

$$0 \le g_n(x) \le^{41)} g_N(x) \le g_{N_{t_i}}(x) < \epsilon$$

This is uniform convergence.
Examples:

1. See need K compact
   $f_n(x) = \frac{1}{nx+1}$ on $K = (0,1]$
   $f_n(x) \to 0^{42)}$ pointwise
   $f_{n+1}(x) \le f_n(x)$
   f_n, f continuous

   $f_n(1/n) = 1/2$ for all n so there does not exist N such that for all $n \ge N$ and for all $x \in (0,1]$, $|f_n(x)| < 1/2$.
2. $f_n(x) = x^n$ on $[0,1]$
   Everything satisfied except continuity of f.
3. $f_n \to 0$ pointwise
   $f_n(1/n) = n$ so convergence is not uniform
   f_n are not decreasing pointwise.

graph of $f_n(x)$: peak of height n at $x = 1/n$

Function Spaces $C(X)$ = continuous functions $f \colon X \to \mathbb{R}$ vector spaces
$C_b(X)$ = continuous, bounded functions $f \colon X \to \mathbb{R}$ subspaces
When X is compact $C(X) = C_b(X)$
$C(\mathbb{R}) \setminus C_b(\mathbb{R})$: $f(x) = x$

Define $\|f\| = \sup_{x \in X} |f(x)|$ when $f \in C_b(X)$
"sup norm" or "uniform" norm (exercise)
$|f(x)| \le \|f\|$ for all $x \in X$
Defines a metric on $C_b(x)$ by $d(f,g) = \|f - g\|$

Ball $B(f,r)$:

figure: g within a ϵ-tube of f

Take $f_n, f \in C_n(X)$
Recall $f_n \to f$ uniformly means for all $\epsilon > 0$ there exists N such that $|f_n(x) - f(x)| \le \epsilon$ for all $n \ge N$ and for all $x \in X$.

figure: $d(f,g) = \|f - g\|$

$$\begin{aligned} &\iff \sup_{x \in X} |f_n(x) - f(x)| \le \epsilon \quad \forall n \ge N \\ &\iff \|f_n - f\| \le \epsilon \quad \forall n \ge N \\ &\iff d(f_n, f) \le \epsilon \quad \forall n \ge N \\ &\iff f_n \to f \text{ in metric space } C_b(x) \end{aligned}$$

41) by g_n decreasing
42) $= f$

$\{f_n\}$ in $C_b(x)$ is Cauchy if and only if $\{f_n\}$ is uniformly Cauchy

Theorem: $C_b(X)$ is a complete metric space
Proof: Suppose $\{f_n\}$ in $C_b(X)$ is a Cauchy sequence. Then $\{f_n\}$ is uniformly Cauchy and so it converges uniformly to some $f \in C(X)$.
Get N such that $|f(x) - F_N(x)| \leq 1$ for all $x \in X$

$$\begin{aligned} &\implies |f(x)| \leq 1 + |f_N(x)| \leq 1 + \|f_N\| \\ &\implies \|f\| = \sup_{x \in X} |f(x)| \leq 1 + \|f_N\| < \infty \\ &\implies f \in C_b(X) \end{aligned}$$

Hence $f_n \to f$ in uniform norm.
Therefore $C_b(X)$ is complete.

$C_b(X)$ is a *complete normed vector space*, i.e., a *Banach space.*

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$C(X)$, $C_b(X)$
$\|f\| = \sup_{x \in X} |f(x)|$ for any $f \in C_b(X)$
$d(f, g) = \|f - g\|$
$(C_b(X), d)$ is a complete metric space

figure: ϵ-tube around f

1. **Example of an open set in $C[0,1]$**

$$B = \{\, f \in C[0,1] : f(x) > 0 \quad \forall x \in [0,1] \,\}$$

Take $\epsilon = \inf_{x \in [0,1]} f(x)$, > 0 by E.V.T.
If $g \in B(f, \epsilon) \iff |g(x) - f(x)| < \epsilon \quad \forall x \in [0,1]$

$$\begin{aligned} \implies g(x) &> f(x) - \epsilon \qquad \forall x \in [0,1] \\ &\geq \inf f - \epsilon \implies g \in B \end{aligned}$$

2.

$$C = \{\, f \in C_b(\mathbb{R}) : f(x) > 0 \quad \forall x \,\}$$

Claim: If $f \in C$ and $\inf_{x \in \mathbb{R}} f = 0$ then f is not an interior point of C. (e.g., $f(x) = \frac{1}{|x|+1}$)
Take any $\epsilon > 0$. Take $g = f - \frac{\epsilon}{2} \in B(f, \epsilon)$
Choose any x such that $f(x) < \frac{\epsilon}{2}$ and then $g(x) < 0$ so $g \notin C$.

3.

$$D = \{\, f \in C_b(\mathbb{R}) : f(x) \leq 0 \quad \forall x \,\}$$

Claim: D is closed.
Let $f_n \in D$ and suppose $f_n \to f$, i.e., $f_n \to f$ uniformly.
But then $f_n \to f$ pointwise. So if $f_n \leq 0$ at every x then $f(x) \leq 0 \quad \forall x$ so $f \in D$.

Compactness in $C_b(X)$
Compact $\implies$ closed and bounded
$E \subset C_b(X)$ is bounded means $\exists f \in C_b(X)$ and M constant such that $E \subseteq B(f, M)$
Then $E \subseteq B(0, M + \|f\|)$ because if $g \in B(f, M)$ then $\|g\| \leq \|g - f\| + \|f\| < M + \|f\| \implies B(f, M) \subseteq B(0, \|f\| + M)$

- call this *uniformly bounded*

Restate: E is bounded iff $\exists M_0$ such that $\|f\| \leq M_0 \quad \forall f \in E$
Example: In $C[0,1]$ closed and bounded $\not\Longrightarrow$ compact.

$$E = \left\{ f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2} : n = 1, 2, 3, \ldots \right\}$$

If $f \in E$, then $0 \le f(x) \le 1\ \forall x$ so $E \subseteq B(0, 1+\epsilon)$.
So E is bounded.
Closed? Say g is an accumulation point of E.
Get $f_{n_k} \to g$ with $f_{n_k} \in E$, $n_1 < n_2 < \cdots$
$f_{n_k} = \frac{x^2}{x^2+(1-n_k x)^2} \to 0$ pointwise.
Look at $f_{n_k}(\frac{1}{n_k}) = 1$ so $\sup_x |f_{n_k} - 0|$[43] $= 1\ \forall n_k$
Thus $f_{n_k} \not\to 0$ uniformly.
Hence there is no accumulation point g.
In fact, no subsequence of (f_n) converges uniformly.
Hence E is closed as it has no accumulation points and E is not compact because fails B–W characterization of compactness.

Equicontinuity
Definition: Let $E \subseteq C(X)$. We say E is *equicontinuous* if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall f \in E$ and $\forall x, y \in X$ such that $d(x,y) < \delta$, we have $|f(x) - f(y)| < \epsilon$.
If $E = \{f\}$ then equicontinuity is uniform continuity.
If $E = \{f_1, \ldots, f_n\}$ then E is equicontinuous if and only if each f_i is uniformly continuous (just take minimum δ that works for $f_1, \ldots, f_n$)
E equiconinuous $\implies$ each $f \in E$ is uniformly continuous.
Not equicontinuous means $\exists \epsilon > 0$ such that $\forall \delta > 0\ \exists f \in E$ and $x, y \in X$ such that $d(x,y) < \delta$ but $|f(x) - f(y)| \ge \epsilon$.

Example:

1. $E = \{\, x^n : n = 1, 2, 3, \ldots \,\} \subseteq C[0,1]$: not equicontinuous
   Take $\epsilon = \frac{1}{2}$ and take any δ. Take $x = 1$, $y = 1 - \frac{\delta}{2}$.
   Pick n so $(1 - \frac{\delta}{2})^n < \frac{1}{2}$.
   Then $|f_n(y$[44]$) - f_n(x$[45]$)| > 1 - \frac{1}{2} = \epsilon$.

   graph of x^n for n large
2. $E = \left\{ f_n(x) = \frac{x^2}{x^2+(1-nx)^2} : n = 1, 2, \ldots \right\}$
   $|f_n(\frac{1}{n}) - f_n(0)| = 1\ \forall n$
   So E is not equicontinuous.
3. $C[0,1]$ is not equicontinuous, since it contains subsets that are not equicontinuous.
4. Fix M. $E = \{\, f \in C[0,1] : |f(x) - f(y)| \le M|x - y| \quad \forall x, y \in [0,1] \,\}$ is equicontinuous.
   Take $\delta = \frac{\epsilon}{M}$.
5. $E_0 = \{\, f \in C[0,1] : |f'(x)| \le M \quad \forall x \in [0,1] \,\} \subseteq E$ (above, in 4.), so it is equicontinuous.

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Equicontinuity
Definition: Say $E \subseteq C(X)$ is *equicontinuous* if $\forall \epsilon > 0\ \exists \delta > 0$ such that if $d(x,y) < \delta$ then $|f(x) - f(y)| < \epsilon\ \forall f \in E$.

Example: $E = \{\, f \in C(\mathbb{R}) : f'$ exists and $|f'(x)| \le M\ \forall x \in X$ and $\forall f \in E \,\}$.
Then E is equicontinuous.
Proof: By Mean Value Theorem $|f(x) - f(y)|$[46] $\le M|x - y|\ \forall x, y$
Given ϵ we take $\delta = \frac{\epsilon}{M}$.

Proposition: If $E \subseteq C(X)$ is equicontinuous then so is $\overline{E}$.
Proof: Let $f \in \overline{E} \setminus E$ and let $\epsilon > 0$.
Get $f_n \in E$ such that $f_n \to f$, i.e., $f_n \to f$ uniformly.

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43) $= \|f_{n_k}\| = 1$
44) $= 1 - \frac{\delta}{2}$
45) $= 1$
46) $= |f'(z)||x - y|$ for some z

So $\exists N$ such that $\|f_N - f\|$[47] $< \epsilon$. Get δ that works for ϵ and E.
Let $x, y \in X$ with $d(x, y) < \delta$, then

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \epsilon + \epsilon + \epsilon = 3\epsilon. \end{aligned}$$

This proves $\overline{E}$ is equicontinuous.

Proposition: Suppose X is compact and $f_n \in C(X)$.
If $f_n \to f$ uniformly, then $E = \{ f_n : n = 1, 2, \dots \}$ is equicontinuous.
f is continuous being uniform limit of continuous functions.
Proof: f is uniformly continuous being continuous on a compact set of X.
Let $\epsilon > 0$. Get δ for f.
Get N such that $\|f_n - f\| < \epsilon$ $\forall n \ge N$.
For any $n \ge N$ and x, y such that $d(x, y) < \delta$,

$$\begin{aligned} |f_n(x) - f_n(y)| &\le |f_n(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_n(y)| \\ &< 3\epsilon \end{aligned}$$

For each f_i, $i = 1, \dots, N-1$ get $\delta_i > 0$ such that $d(x, y) < \delta_i \implies |f_i(x) - f_i(y)| < 3\epsilon$ (can do as each f_i is uniformly continuous)
Take $\delta_0 = \min(\delta, \delta_1, \dots, \delta_{N-1})$.
If $d(x, y) < \delta_0$ then $|f_n(x) - f_n(y)| < 3\epsilon$ $\forall n$.
So E is equicontinuous.

Example: $E = \left\{ f_n(x) = \frac{\sin nx}{\sqrt{n}} : x \in [0, 2\pi] \right\}$
$|f_n(x)| \le \frac{1}{\sqrt{n}} \to 0$ so $f_n \to 0$ uniformly. $\implies E$ is equicontinuous.
But $f_n'(x) = \frac{n \cos nx}{\sqrt{n}} = \sqrt{n} \cos nx$ so $f_n'(0) = \sqrt{n} \to \infty$.

Uniformly Bounded
$E \subseteq C(X)$ is uniformly bounded if $E \subseteq B(0, M)$ for some M, equivalently $\exists M$ such that $\|f\| \le M$ $\forall f \in E$.

Definition: Say $E \subseteq C(X)$ is *pointwise bounded* if $\forall x \in X$ $\exists M_x$ such that $|f(x)| \le M_x$ $\forall f \in E$.

Uniformly bounded $\implies$ pointwise bounded, but not conversely.
Fix $x \ne 0$. Have $f_n(x) \ne 0$ $\forall n \ge N$ where $\frac{1}{N} < x$.

$$\sup |f_n(x)| \le \max(|f_1(x)|, \dots, |f_N(x)|)$$

graph: $f_n(x)$ has peak of n and is zero for $x > \frac{1}{n}$

So $\{f_n\}$ is pointwise bounded, but not uniformly bounded.

Proposition: If X is compact and E is equicontinuous and pointwise bounded, then E is uniformly bounded.
Proof: Take $\epsilon = 1$. Get δ by equicontinuity so $d(x, y) < \delta \implies |f(x) - f(y)| < 1$ $\forall f \in E$
Look at balls $B(x, \delta)$ for $x \in X$. This is an open cover of compact X so take a finite subcover, say $B(x_1, \delta), \dots, B(x_n, \delta)$.
Let $M_i = \sup\{ |f(x_i)| : f \in E \}$ ($< \infty$ by pointwise boundedness of E)
Take $M = (\max_{i=1,\dots,n} M_i) + 1$.
Let $x \in X$. There is a ball $B(x_i, \delta)$ containing x.

$$\begin{aligned} \implies d(x, x_i) < \delta \implies |f(x)| &\le |f(x) - f(x_i)| + |f(x_i)| \\ &\le 1 + M_i \\ &\le M \end{aligned}$$

Theorem: Let X be compact. Let $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$ be a pointwise bounded, equicontinuous family. Then

[47] $= \sup_{x \in X} |f_N(x) - f(x)|$

(1) $\{f_n\}$ is uniformly bounded. (already done)

(2) There is a subsequence of the sequence (f_n) which converges uniformly.

Corollary: (Arzela–Ascoli Theorem)
Let X be compact. $E \subseteq C(X)$ is compact if and only if E is pointwise (uniformly) bounded, closed and equicontinuous.
Proof: ($\Longrightarrow$) E compact $\implies$ E bounded (meaning uniformly bounded) and closed
Suppose E is not equicontinuous. This means $\exists \epsilon > 0$ such that $\forall \delta = \frac{1}{n}$ there are $x_n, y_n \in X$ with $d(x_n, y_n) < \frac{1}{n}$ and $\exists f_n \in E$ with $|f_n(x_n) - f_n(y_n)| \geq \epsilon$[48].
Since E is compact the Bolzano–Weierstrass characterization of compactness says there is a subsequence $f_{n_k} \to$[49] $f \in E$.
Hence the set $\{f_{n_k}\}$ is equicontinuous and hence $\exists \delta_0$ such that $d(x,y) < \delta_0 \implies |f_{n_k}(x) - f_{n_k}(y)| <$[50] ϵ $\forall n_k$.
Take n_k such that $\delta_0 > \frac{1}{n_k}$ so $d(x_{n_k}, y_{n_k}) < \frac{1}{n_k} < \delta_0$ so $|f_{n_k}(x_{n_k}) - f_{n_k}(y_{n_k})| < \epsilon$ by (1) and this contradicts (2).

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Theorem: X compact. $\{f_n\} \subseteq C(X)$ be a pointwise bounded and equicontinuous set. Then

(a) $\{f_n\}$ uniformly bounded

(b) there exists a subsequence of $\{f_n\}$ which converges uniformly

Corollary: (Arzela–Ascoli Theorem): For X compact, $E \subseteq C(X)$ is compact if and only if E is pointwise bounded, closed and equicontinuous.
Proof: ($\Longleftarrow$) Let $\{f_n\}$ be a sequence in E.
Since E is pointwise bounded and equicontinuous, the same is true for $\{f_n\}$. By theorem there exists a uniformly convergent subsequence and the limit must belong to E since E is closed. By Bolzano–Weierstrass characterization of compactness this implies E is compact.

Lemma 1: Let K be a countable set. Let $f_n \colon K \to \mathbb{R}$, $n = 1, 2, \ldots$ be a pointwise bounded family. There there exists subsequence (g_n) of (f_n) which converges pointwise.
Proof: Let $K = \{x_1, x_2, x_3, \ldots\}$.
Start by looking at $\{f_n(x_1)\}_{n=1}^{\infty}$.
Since $\{f_n\}$ are pointwise bounded, the sequence $\{f_n(x_1)\}$ is a bounded sequence of real numbers and so by Bolzano–Weierstrass there exists a convergent subsequence, say $f_{1,1}(x_1), f_{1,2}(x_1), \ldots$.
Thus $\{f_{1,n}\}_{n=1}^{\infty}$ is a subsequence of $\{f_n\}$ converging at x_1.
Look at $\{f_{1,n}(x_2)\}_{n=1}^{\infty}$: bounded sequence of real numbers therefore convergent subsequence, say $f_{2,1}(x_2), f_{2,2}(x_2), \ldots$.

$$\begin{array}{cccccccl}
f_1 & f_2 & f_3 & f_4 & \cdots & f_k & & \\
\boxed{f_{11}} & f_{12} & f_{13} & f_{14} & \cdots & f_{1k} & & \text{converges at } x_1 \\
f_{21} & \boxed{f_{22}} & f_{23} & f_{24} & \cdots & f_{2k} & & \text{converges at } x_1, x_2 \\
f_{31} & f_{32} & \boxed{f_{33}} & f_{34} & \cdots & f_{3k} & & \text{converges at } x_1, x_2, x_3 \\
\vdots & & & & & & & \\
f_{k1} & f_{k2} & f_{k3} & f_{k4} & \cdots & \boxed{f_{kk}} & & \text{converges at } x_1, x_2, \ldots, x_k
\end{array}$$

In general, given $(f_{k,n})$ a subsequence of (f_n) which converges at $x_1, x_2, \ldots, x_k$, consider $(f_{k,n}(x_{k+1}))$:
Get a convergent subsequence $(f_{k+1,n}(x_{k+1}))$. So $(f_{k+1,n})$ converges at $x_1, x_2, \ldots, x_{k+1}$.
Put $g_n = f_{n,n}$. (g_n) is a subsequence of (f_n).

[48] (2)
[49] uniform convergence
[50] (1)

Furthermore $(g_n)_{n=k}^{\infty}$ is a subsequence of $(f_{k,n})$ and hence converges at x_k.
So (g_n) converges pointwise on K.

Lemma 2: Any compact metric space X is separable (i.e., countable dense set)
Proof: For each n, the balls $B(x, \frac{1}{n})$, $x \in X$ cover X. Get a finite subcover $B(x_{n,1}, \frac{1}{n}), \ldots, B(x_{n,k_n}, \frac{1}{n})$.
Put $K_n = \{x_{n,1}, \ldots, x_{n,k_n}\}$ and $K = \bigcup_{n=1}^{\infty} K_n$: K is countable.
Given $y \in X$ and $\epsilon > 0$. Take n such that $\frac{1}{n} < \epsilon$. Have $y \in B(x_{n,j}, \frac{1}{n})$ for some j.
Therefore $x_{n,j} \in B(y, \frac{1}{n}) \subset B(y, \epsilon)$, so $y \in \overline{K}$, therefore K is dense.
Proof of Theorem (b): Let K be a countable dense set on X.
Think about $f_n \colon K \to \mathbb{R}$: Pointwise bounded.
By Lemma 1 there exists a pointwise convergent (on K) subsequence (g_n).
We'll prove (g_n) converges uniformly on all of X.
Suffices to prove (g_n) is uniformly Cauchy.
Take $\epsilon > 0$. Find N such that $\forall n, m \geq N$,

$$|g_n(x) - g_m(x)| < \epsilon \qquad \forall x \in X.$$

By equicontinuity $\exists \delta > 0$ such that

$$d(x, y) < \delta \implies |g_n(x) - g_n(y)| < \epsilon \qquad \forall n.$$

Notice balls $B(x, \delta)$, $x \in K$ cover X because K is dense. By compactness of X, $\exists x_1, \ldots, x_M$ such that $\bigcup_1^M B(x_i, \delta)$ covers X.
If $y \in X$ then $y \in B(x_i, \delta)$ for some x_i.
By choice of δ, $|g_n(y) - g_n(x_i)| < \epsilon \ \forall n$.
$\{g_n(x_i)\}$ converges for each i and so is Cauchy.
Hence $\exists N_i$ such that if $n, m \geq N$, then $|g_n(x_i) - g_m(x_i)| < \epsilon$ (2).
Let $N = \max(N_1, \ldots, N_M)$.
Let $y \in X$ and $n, m \geq N$. Get i such that $y \in B(x_i, \delta)$ so

$$|g_k(y) - g_k(x_i)| < \epsilon \qquad \forall k. \tag{1}$$

$$\begin{aligned}|g_n(y) - g_m(y)| &\leq |g_n(y) - g_n(x_i)| + |g_n(x_i) - g_m(x_i)| + |g_m(x_i) - g_m(y)| \\ &< \epsilon^{51)} + \epsilon^{52)} + \epsilon^{53)} = 3\epsilon\end{aligned}$$

Therefore (g_n) is uniformly Cauchy.

PMATH 351 Lecture 23: November 9, 2009

Taylor Series
$\exists f \in C^{\infty}$ where Taylor polynomials do not converge to f.

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$f^{(k)}(0) = 0 \ \forall k$. All Taylor polynomials (centred at 0) are identically 0. So they don't converge to f except at 0.

Inner Product Spaces
$C[0,1]$: Define inner product $\langle f, g \rangle = \int_0^1 fg$.

$$\left.\begin{aligned} \|f\|_2 &= \sqrt{\langle f, f \rangle} = \left(\int_0^1 f^2 \right)^{1/2} \\ d_2(f, g) &= \left(\int_0^1 (f - g) \right)^{1/2} \end{aligned}\right\} L_2$$

51) (1)
52) (2)
53) (1)

- metric on $C[0,1]$
- not complete

Apply Gram Schmidt process to $\{1, x, x^2, \ldots\}$, to get the Legendre polynomials $\{p_n\}$.
Given $f \in C[0,1]$, let $f_N = \sum_{n=1}^{N} \langle f, p_n \rangle p_n$. Then $f_N \to f$ in $\|\cdot\|_2$. (PMATH 354!)

Example: $f(x) = \sqrt{x}$ on $[0,1]$. Put $p_1(t) = 0$, $p_{n+1}(t) = p_n(t) + \frac{1}{2}(t - p_n^2(t))$
Claim: $p_n \to f$ uniformly.

$$p_2(t) = 0 + \tfrac{1}{2}(t - 0) = \tfrac{1}{2}t$$
$$p_3(t) = \tfrac{1}{2}t + \tfrac{1}{2}(t - \tfrac{1}{4}t^2)$$

Show $p_n \to f$ pointwise

$$p_n(t) \leq p_{n+1}(t) \qquad \forall n, t$$

Show p_n, f are continuous. Dini's theorem implies $p_n \to f$ uniformly.
Proceed by induction. Assume $0 \leq p_1(t) \leq p_2(t) \leq \cdots \leq p_n(t) \leq \sqrt{t}$.
$n = 1$: Free.

$$\begin{aligned} \sqrt{t} - p_{n+1}(t) &= \sqrt{t} - (p_n(t) + \tfrac{1}{2}(t - p_n^2(t))) \\ &= \sqrt{t} - p_n(t) - \tfrac{1}{2}(\sqrt{t} - p_n(t))(\sqrt{t} + p_n(t)) \\ &= (\sqrt{t} - p_n(t))(1 - \tfrac{1}{2}(\sqrt{t} + p_n(t))) \end{aligned}$$

But $p_n(t) \leq \sqrt{t}$, so $\geq (\sqrt{t} - p_n(t))(1 - \sqrt{t}) \geq 0$.
$\implies p_{n+1}(t) \leq \sqrt{t}$, $p_{n+1}(t) = p_n(t) + \frac{1}{2}(t - p_n^2(t))^{54)}$
so $p_{n+1}(t) \geq p_n(t)$.

So $\{p_n(t)\}$ is increasing and bounded above for fixed $t \in [0,1]$, hence it converges by Bolzano–Weierstrass, say $\{p_n(t)\} \to f(t)$ (pointwise convergence)

$$p_{n+1}(t) = p_n(t) + \tfrac{1}{2}(t - p_n^2(t))$$
$$f(t) = f(t) + \tfrac{1}{2}(t - f^2(t)) \implies t = f^2(t), \text{ so } \sqrt{t} = f(t)$$

By Dini's theorem convergence is uniform.

Weierstrass Theorem: Let $f\colon [0,1] \to \mathbb{R}$ be continuous and let $\epsilon > 0$. Then there exists a polynomial p such that $\|f - p\| < \epsilon$.
In fact, the Bernstein polynomials

$$p_n(f) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

converge uniformly to f.

Intuitive Identity: Toss a coin n times; probability of heads x, probability of tails $1 - x$. Probability of k heads in n tosses:

$$\binom{n}{k} x^k (1-x)^{n-k}$$

Suppose pay $f(\frac{k}{n})$ dollars for k heads in n tosses. Expected pay off over n tosses: $\sum_{k=0}^{n} \binom{n}{k} f(\frac{k}{n}) x^k (1-x)^{n-k} = p_n(x)$.

In long run we expect xn heads in n tosses, so expect pay off of $f(\frac{xn}{n}) = f(x)$. So intuitively $p_n(x) \to f(x)$.

Proof of Theorem: Technical Calculations:

(1) $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$. Differentiate with respect to x, leave y fixed.

$f(x,y) = (x+y)^n$
$\frac{\partial f}{\partial x}(x,y) = n(x+y)^{n-1}$

(2) $n(x+y)^{n-1} = \sum_{k=0}^n \binom{n}{k} k x^{k-1} y^{n-k}$

(3) $n(n-1)(x+y)^{n-2} = \sum_{k=0}^n \binom{n}{k} k(k-1) x^{k-2} y^{n-k}$

(2′) $x \cdot (2)$: $nx(x+y)^{n-1} = \sum_{k=0}^n \binom{n}{k} k x^k y^{n-k}$

(3′) $x^2 \cdot (3)$: $n(n-1)x^2(x+y)^{n-2} = \sum_{k=0}^n \binom{n}{k} k(k-1) x^k y^{n-k}$

Put $r_k(x) = \binom{n}{k} x^k (1-x)^{n-k}$
$p_n(x) = \sum_{k=0}^n f(\frac{k}{n}) r_k(x)$
Take $y = 1-x$

(1) $1 = \sum_{k=0}^n r_k(x)$

(2′) $nx = \sum_{k=0}^n k r_k(x)$

(3′) $n(n-1)x^2 = \sum_{k=0}^n k(k-1) r_k(x) = \sum k^2 r_k(x) - \sum k r_k(x) = \sum_{k=0}^n k^2 r_k(x) - nx$

$$\begin{aligned}\sum (k-nx)^2 r_k(x) &= \sum k^2 r_k(x) - 2\sum nkx r_k(x) + \sum (nx)^2 r_k(x) \\ &= n(n-1)^2 x^2 + nx - 2nxnx + (nx)^2\end{aligned}$$

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Weierstrass Theorem
Polynomials are dense in $C[0,1]$.

$$\text{i.e., } \forall f \in C[0,1] \text{ and } \forall \epsilon > 0 \text{ there exists polynomial } p \\ \text{such that } \|f-p\| = \sup_{x\in[0,1]} |f(x) - p(x)| < \epsilon$$

Bernstein Proof
Show $p_n(x) = \sum_{k=0}^n \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$ converges uniformly to f.

(1) $\sum_{k=0}^n r_k(x) = 1$ where $r_k(x) = \binom{n}{k} x^k (1-x)^{n-k}$

(2) $\sum_{k=0}^n (k-nx)^2 r_k(x) = nx(1-x)$

Let $f \in C[0,1]$, say $|f(x)| \le M \ \forall x \in [0,1]$
Also f is uniformly continuous, so given $\epsilon > 0$ get $\delta > 0$ such that $|x-y| < \delta \implies |f(x) - f(y)| < \epsilon$
Take N such that $\frac{2M}{\delta^2 N} < \epsilon$.
Let $n \ge N$. Fix $x \in [0,1]$.

$$\begin{aligned}|p_n(x) - f(x)| &= \left|\sum_{k=0}^n f(\tfrac{k}{n}) r_k(x) - f(x) \sum_{k=0}^n r_k(x)\right| \\ &= \left|\sum_{k=0}^n \big(f(\tfrac{k}{n}) - f(x)\big) r_k(x)\right|\end{aligned}$$

[54] ≥ 0 by induction assumption

Divide ks into 2 classes

$$\begin{aligned}
A &= \left\{\, k : |\tfrac{k}{n} - x| < \delta \iff |k - nx| < \delta n \,\right\} \\
B &= \left\{\, k : |\tfrac{k}{n} - x| \geq \delta \iff |k - nx| \geq \delta n \,\right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=0}^{n} |f(\tfrac{k}{n}) - f(x)| r_k(x) \\
&\leq \sum_{k\in A} |f(\tfrac{k}{n}) - f(x)| r_k(x) + \sum_{k\in B} |f(\tfrac{k}{n}) - f(x)| r_k(x) \\
&\leq \sum_{k\in A} \epsilon r_k(x) + \sum_{|k-nx|\geq\delta n} 2M r_k(x) \frac{(k-nx)^2}{(k-nx)^2} \\
&\leq \sum_{k\in A} \epsilon r_k(x)^{55)} + \sum_{k=0}^{n} \frac{2M r_k(x)(k-nx)^2}{(\delta n)^2} \\
&\leq \epsilon + \frac{2M}{(\delta n)^2} nx(1-x) \qquad \text{by (2)} \\
&= \epsilon + \frac{2M}{\delta^2} \cdot \frac{1}{n} \leq \epsilon + \frac{2M}{\delta^2 N} < 2\epsilon
\end{aligned}$$

This shows $\|p_n - f\| \leq 2\epsilon \ \forall n \geq N$
i.e., $p_n \to f$ uniformly.

Approximation by trigonometric polynomials

$$\sum_{n=0}^{N} a_n \sin nx + b_n \cos nx = \sum_{n=-N}^{N} c_n e^{inx}$$

$a_n, b_n \in \mathbb{C}$, $c_n \in \mathbb{C}$

$z \in \mathbb{C}$
$|z| = 1$
$z = e^{ix}$

$$\begin{aligned}
e^{ixn} &= \cos xn + i \sin xn \\
\frac{e^{ixn} + e^{-ixn}}{2} &= \cos xn \\
\frac{e^{ixn} - e^{-ixn}}{2i} &= \sin xn
\end{aligned}$$

- uniformly approximate continuous, 2π periodic functions
  $= C[0, 2\pi]$ with $f(0) = f(2\pi)$

Inner product spaces:

$$\langle f, g\rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)}\,\mathrm{d}x$$

$$\|f\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \,\mathrm{d}x \right)^{1/2}$$

$\{e^{inx}\}_{n=-\infty}^{\infty}$ are orthonormal

55) $= \epsilon$

Check:

$$\begin{aligned}\langle e^{inx}, e^{imx}\rangle &= \frac{1}{2\pi}\int_0^{2\pi} e^{inx}e^{-imx}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)x}\,\mathrm{d}x \\ &=^{56)}\ \frac{1}{2\pi}\frac{e^{i(n-m)x}}{i(n-m)}\bigg|_0^{2\pi} \\ &= 0\end{aligned}$$

"Best" approximation (in inner product sense) to f from

$$\operatorname{span}\{\, e^{inx} : n = -N, \dots, N \,\} = \sum_{n=-N}^{N} \langle f, e^{-inx}\rangle e^{inx} = \sum_{n=-N}^{N} \hat{f}(n)e^{inx} = f_N$$

$$\begin{aligned}\langle f, e^{inx}\rangle &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,\mathrm{d}x \\ &\equiv \hat{f}(n)^{57)}\end{aligned}$$

Big Theorem (PM354)
$f_N \to f$ in $\|\cdot\|_2$
i.e., $\left(\frac{1}{2\pi}\int_0^{2\pi}|f_N - f|^2\right)^{1/2} \to 0$
This does not even guarantee pointwise convergence (Big Theorem PM354).

Let $K_n(t)^{58)} = \sum_{j=-n}^{n}\left(1 - \frac{|j|}{n+1}\right)e^{ijt}$.
Put $f_n(x) = \frac{1}{2\pi}\int_0^{2\pi} K_n(t)f(x-t)\,\mathrm{d}t = K_n * f(x)$

Theorem: $f_n \to f$ uniformly and f_n are trigonometric polynomials
First, show f_n are trigonometric polynomials:

$$\begin{aligned}f_n(x) &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{j=-n}^{n}\left(1 - \frac{|j|}{n+1}\right)e^{ijt}f(x-t)\,\mathrm{d}t \\ &= \frac{1}{2\pi}\sum_{j=-n}^{n}\left(1 - \frac{|j|}{n+1}\right)\int_0^{2\pi} e^{ijt}f(x-t)\,\mathrm{d}t\end{aligned}$$

Change of variable: Let $u = x - t$, $\mathrm{d}t = \mathrm{d}u$

$$\begin{aligned}&= \frac{1}{2\pi}\sum_{j=-n}^{n}\left(1 - \frac{|j|}{n+1}\right)\underbrace{\int_0^{2\pi} e^{ij(x-u)}f(u)\,\mathrm{d}u}_{\int_0^{2\pi} e^{ijx}e^{-iju}f(u)\,\mathrm{d}u} \\ &= \sum_{-n}^{n}\left(1 - \frac{|j|}{n+1}\right)e^{ijx}\underbrace{\left(\frac{1}{2\pi}\int_0^{2\pi} e^{-iju}f(u)\,\mathrm{d}u\right)}_{=\hat{f}(j)} \\ &= \sum_{j=-n}^{n}\underbrace{\left(1 - \frac{|j|}{n+1}\right)\hat{f}(j)}_{=c_j}\,e^{ijx}\end{aligned}$$

---

56) if $n \neq m$
57) nth Fourier coefficients of f
58) Fejer's kernel

So f_n is a trigonometric polynomial of degree $\leq n$.

$$\begin{aligned}\hat{f}_n(j) &= \left(1 - \frac{|j|}{n+1}\right)\hat{f}(j) \\ &= \hat{K}_n(j)\hat{f}(j)\end{aligned}$$

so, $f_n(x) = \sum_{j=-n}^{n}\left(1 - \frac{|j|}{n+1}\right)\hat{f}(j)e^{ijx}$

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Theorem: Trigonometric polynomials are uniformly dense in 2π-periodic, continuous functions.

Given f continuous and 2π periodic define

$$f_n(t) = \sum_{j=-n}^{n} \hat{f}(j)^{59)}\left(1 - \frac{|j|}{n+1}\right)e^{ijt}$$

Then $f_n \to f$ uniformly.

$$\begin{aligned}\text{Also } f_n(x) &= \frac{1}{2\pi}\int_0^{2\pi} f(x-t)K_n(t)\,\mathrm{d}t \\ \text{where } {K_n}^{60)}(t) &= \sum_{j=-n}^{n}\left(1 - \frac{|j|}{n+1}\right)e^{ijt}\end{aligned}$$

Sketch of Proof

(1) $\frac{1}{2\pi}\int_0^{2\pi} K_n(t)\,\mathrm{d}t = \frac{1}{2\pi}\sum_{j=-n}^{n}\left(1 - \frac{|j|}{n+1}\right)\int_0^{2\pi} e^{ijt}\,\mathrm{d}t = 1$

(2) $K_n(t) = \frac{1}{n+1}\frac{\sin^2(\frac{n+1}{2})t}{\sin^2\frac{t}{2}} \geq 0$

figure: functions approximation Dirac's delta

(3) If fix $\delta > 0$ and let $\delta < t < 2\pi - \delta$ then $K_n(t) \leq \frac{1}{n+1}c(\delta) \to 0$ as $n \to \infty$. Fix δ.

$$\begin{aligned}\frac{1}{2\pi}\int_\delta^{2\pi-\delta} K_n(t)\,\mathrm{d}t &\leq \frac{1}{2\pi}\int_\delta^{2\pi-\delta}\frac{c(\delta)}{n+1}\,\mathrm{d}t \\ &\leq \frac{c(\delta)}{n+1} \to 0 \text{ as } n \to \infty\end{aligned}$$

$$\begin{aligned}|f_n(x) - f(x)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} f(x-t)K_n(t)\,\mathrm{d}t - f(x)\right| \\ &\leq \left|\frac{1}{2\pi}\int_0^{2\pi}(f(x-t) - f(x))K_n(t)\,\mathrm{d}t\right| \qquad \text{(by (1))} \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}|(f(x-t) - f(x))|K_n(t)\,\mathrm{d}t\end{aligned}$$

Fix $\epsilon > 0$. Pick $\delta > 0$ by uniform continuity so $|t| < \delta \implies |f(x-t) - f(x)| < \epsilon$.
Get M such that $|f(x)| < M\ \forall x$.

$$\frac{1}{2\pi}\left(\int_0^\delta(1) + \int_{2\pi-\delta}^{2\pi}(2) + \int_\delta^{2\pi-\delta}(3)\right) \leq \epsilon + \epsilon + \epsilon = 3\epsilon \qquad \forall n \geq N$$

$$(3) \qquad \leq \int_\delta^{2\pi-\delta} 2MK_n(t)\,\mathrm{d}t \leq 2M\frac{c(\delta)}{n+1} < \epsilon$$

59) $\langle f, e^{ijx}\rangle$

60) Feijer kernel

if $n \geq N$ where $\frac{2Mc(\delta)}{N} < \epsilon$

$$(1) \qquad \leq \frac{1}{2\pi}\int_0^\delta \epsilon K_n(t)\,\mathrm{d}t \leq \frac{1}{2\pi}\int_0^{2\pi} \epsilon K_n(t)\,\mathrm{d}t = \epsilon$$

(2) $t = 2\pi - u$ where $u \in [0,\delta]$ when $t \in [2\pi - \delta, 2\pi]$

$$\frac{1}{2\pi}\int_0^\delta |f(x - 2\pi + u)^{61)} - f(x)|K_n(2\pi - u)\,\mathrm{d}u \leq \frac{1}{2\pi}\int_0^\delta \epsilon K_n(2\pi - u)\,\mathrm{d}u \leq \epsilon$$

$|-u| \leq \delta$

Thus $f_n \to f$ uniformly.

Stone–Weierstrass Theorem

Terminology: A family $\mathcal{A}$ of functions (on X) is called an *algebra* if $f, g \in \mathcal{A} \implies f + g \in \mathcal{A}$, $fg \in \mathcal{A}$, $cf \in \mathcal{A}$ for all scalars c

Examples: Polynomials, $C(X)$, Differentiable functions on $\mathbb{R}$.

Say $\mathcal{A}$ *separates points* if $\forall x \neq y \in X$ then $\exists f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Example: polynomials on $[0,1]$

$C(X)$ separates points: $f(z) = d(x,z)$, continuous function, $f(x) = 0$, but $f(y) = d(x,y) \neq 0$ if $x \neq y$.

Stone–Weierstrass Theorem: Let X be compact and let $\mathcal{A} \subseteq C(X)$ be an algebra that separates points. Assume constant functions belong to $\mathcal{A}$. Then $\mathcal{A}$ is dense in $C(X)$.

$$\text{i.e., } \forall \epsilon > 0 \ \&\ \forall f \in C(X)\ \exists g \in \mathcal{A} \text{ such that } \|g - f\| < \epsilon.$$

Corollary: Polynomials are dense in $C[0,1]$.

Separation of points is necessary for $\mathcal{A}$ to be dense

If $\exists x \neq y$ such that $f(x) = f(y)\ \forall f \in \mathcal{A}$ then if $f_n \in \mathcal{A}$ and $f_n \to g$ uniformly, we must have $g(x) = g(y)$. But $\exists g \in C(X)$ such that $g(x) \neq g(y)$

Lemma 1: Suppose B is any algebra $\subseteq C(X)$ containing all constant functions. If $f \in B$, then $|f| \in \overline{B}$.

Proof: Let $c = \|f\| > 0$. We know there exists polynomials p_n such that $p_n \to \sqrt{x}$ uniformly on $[0,1]$.

Suppose $g \in B$, $0 \leq g(x) \leq 1\ \forall x \in X$.

Then $p_n \circ g(x)^{62)}$ is defined $\forall x \in X$.

If $p_n(t) = a_k^{(n)} t^k + \cdots + a_1^{(n)} t + a_0^{(n)}$ then $p_n \circ g(x) = a_k^{(n)} g(x)^k + \cdots + a_1^{(n)} g(x) + a_0^{(n)}$

Also $f \in B$ so $\frac{f^2}{c^2} \in B$ and $0 \leq \frac{f^2}{c^2} \leq 1$.

Therefore $p_n \circ \left(\frac{f^2}{c^2}\right) \in B$.

Know $\forall \epsilon > 0\ \exists N$ such that $|p_n(t) - \sqrt{t}| < \epsilon\ \forall t \in [0,1]$ and $\forall n \geq N$

So $\forall x \in X$

$$\Bigg|\underbrace{p_n\left(\frac{f^2(x)}{c^2}\right)}_{=f_n(x)} - \sqrt{\frac{f^2(x)}{c^2}}^{63)}\Bigg| < \epsilon$$

$\implies \|f_n - \frac{|f|}{c}\| \leq \epsilon\ \forall n \geq N$

$f_n \in B$ and $f_n \to \frac{|f|}{c}$ uniformly

Exercise: $\underbrace{cf_n}_{\in B} \to |f|$ uniformly $\implies |f| \in \overline{B}$

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61) $= f(x - (-u))$

62) $= p_n(g(x))$

63) $= \frac{|f(x)|}{c}$

Stone–Weierstrass Theorem
Algebra $\mathcal{A}$: $f, g \in \mathcal{A} \implies f + g \in \mathcal{A}$
$fg \in \mathcal{A}$
$cf \in \mathcal{A}$
$\mathcal{A} \subseteq C(X, F)$, $F = \mathbb{R}$ or $\mathbb{C}$ separates points
if whenever $x \neq y \in X$
$\exists f \in \mathcal{A}$ such that $f(x) \neq f(y)$

Let X be compact, metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra that separates points. Assume constant functions belong to $\mathcal{A}$. Then $\mathcal{A}$ is dense in $C(X)$.

Lemma 1: Suppose B an algebra $\subseteq C(X)$ that contains the constants. If $f \in B$ then $|f| \in \overline{B}$.
Lemma 2: If $f, g \in \overline{\mathcal{A}}$ then $\max(f, g)$[64] and $\min(f, g) \in \overline{\mathcal{A}}$
Proof: First check $\mathcal{A}$ is an algebra.
Let $f, g \in \overline{\mathcal{A}}$, say f_n[65] $\to f$, g_n[65] $\to g$, $f_n + g_n \in \mathcal{A}$ since $\mathcal{A}$ is an algebra.

$$\begin{matrix} f_n + g_n \to f + g \\ c^{65)} f_n \to cf \end{matrix} \implies \begin{matrix} f + g \in \overline{\mathcal{A}} \\ cf \in \overline{\mathcal{A}} \end{matrix}$$

By Lemma, $|f - g| \in \overline{\mathcal{A}}$.

$$\max(f, g) = \tfrac{1}{2}(f + g + |f - g|) \in \overline{\mathcal{A}}$$
$$\min(f, g) = \tfrac{1}{2}(f + g + |f - g|) \in \overline{\mathcal{A}}$$

Lemma 3: Given $x \neq y \in X$, $a, b \in \mathbb{R}$, there exists $f \in \mathcal{A}$ such that $f(x) = a$, $f(y) = b$
Proof: Since $\mathcal{A}$ separates points there exists $g \in \mathcal{A}$ such that $g(x) \neq g(y)$

$$\text{Put } f(t^{66)}) = a + (b - a)\left(\frac{g(t) - g(x)^{67)}}{\underbrace{g(y) - g(x)}_{\neq 0}}\right)$$
$$= \alpha_1 + \alpha_2 g(t) \in \mathcal{A}$$
$$f(x) = a,\ f(y) = b\ \checkmark$$

Lemma 4: If $f \in C(X)$, $x_0 \in X$ and $\epsilon > 0$ then there exists g[68] $\in \overline{\mathcal{A}}$ such that $g(x_0) = f(x_0)$ and $g(z) \leq f(z) + \epsilon\ \forall z \in X$
Proof: Apply lemma 3 with $x = x_0$, y fixed[69] but arbitrary, $a = f(x_0)$, $b = f(y)$.
Get $h_y \in \mathcal{A}$ such that $h_y(x_0) = f(x_0)$, $h_y(y) = f(y)$.
If $y = x_0$ just take $h_{x_0}(t) = f(x_0)$ (constant function)
Look at $(h_y - f)(y) = 0$.
$h_y - f$ is continuous so $\exists \delta y > 0$ such that $|h_y(z) - f(z)| < \epsilon$ if $d(y, z) < \delta_y$.
Look at balls $\{\, B(y, \delta_y) : y \in X \,\}$: open cover of compact set X, so there is a finite subcover, say

$$B(y_1, \delta y_1), \dots, B(y_k, \delta y_k)$$

Take $g = \min(h_{y_1}, \dots, h_{y_k}) \in \overline{\mathcal{A}}$ by lemma 2.
$g(x_0) = f(x_0)$ as all $h_y(x_0) = f(x_0)$.
If $z \in X$, then $z \in B(y_j, \delta_{y_j})$ for some j
$\implies d(y_j, z) < \delta_{y_j}$
By definition of δ_{y_j}, this implies $h_{y_j}(z) < f(z) + \epsilon$

[64] $= h$, $h(x) = \max(f(x), g(x))$
[65] $\in \mathcal{A}$
[66] $\in X$
[67] $\in \mathbb{R}$
[68] $= g(x_0, \epsilon)$
[69] $y \neq x_0$

$\implies g(z) \le h_{y_j}(z) < f(z) + \epsilon$
Lemma 5: If $f \in C(X)$ and $\epsilon > 0$ there exists $g \in \overline{\mathcal{A}}$ such that $\|g - f\| < \epsilon$.
Proof: For each $x \in X$ by Lemma 4 we get $g_x \in \overline{\mathcal{A}}$ such that $g_x(x) = f(x)$ and

$$g_x(z) \le f(z) + \epsilon \quad \forall z \in X \tag{2}$$

Know $g_x - f(x) = 0$ so there exists $\delta_x > 0$ such that

$$d(x, z) < \delta_x \implies |g_x(z) - f(z)| < \epsilon$$

Balls $B(x, \delta_x)$: $x \in X$ open cover of X
Take a finite subcover, say $B(x_1, \delta_{x_1}), \dots, B(x_L, \delta_{x_L})$
Put $g = \max(g_{x_1}, \dots, g_{x_L}) \in \overline{\mathcal{A}}$
Take $y \in X$ say $y \in B(x_i, \delta_{x_i})$
$\implies |g_{x_i} - f(y)| < \epsilon$

$$f(y) - \epsilon \underset{(1)}{<} g_{x_i}(y) < f(y) + \epsilon$$

$$\begin{aligned} f(y) - \epsilon \underset{(1)}{<} g_{x_i}(y) \le g(y) &= g_{x_j}(y) \text{ (some index)} \\ &\le f(y) + \epsilon \text{ by (2)} \\ \implies |g(y) - f(y)| &\le \epsilon \quad \forall y \in X \\ \implies \|g - f\| &\le \epsilon \end{aligned}$$

Proof of S–W Theorem
Let $f \in C(X)$, and $\epsilon > 0$
By lemma 5 get $g \in \overline{\mathcal{A}}$ such that $\|g - f\| \le \epsilon/2$.
Get $h \in \mathcal{A}$ such that $\|g - h\| \le \epsilon/2$.
By triangle inequality

$$\begin{aligned} \|f - h\| &\le \|f - g\| + \|g - h\| \\ &\le \epsilon \end{aligned}$$

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Complex-Valued Continuous Functions
$\mathbb{C}$ metric space
$d(z, w) = |z - w|$

$$\begin{aligned} |z| = |\operatorname{Re} z + i \operatorname{Im} z| &= \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} \\ &= \|(\operatorname{Re} z, \operatorname{Im} z)\|_{\mathbb{R}^2} \end{aligned}$$

$f\colon X \to \mathbb{C}$
f is continuous at x means whenever

$$\underbrace{x_n \to x}_{\text{converges in } X} \qquad \text{then} \qquad \underbrace{f(x_n) \to f(x)}_{\text{converges in } \mathbb{C}}$$

$f = g + ih$
$f = \operatorname{Re} f + i \operatorname{Im} f$
$\operatorname{Re} f(x) = \operatorname{Re}(f(x))$
$g(x) = \operatorname{Re}(f(x))$
f is continuous iff $\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous where $\operatorname{Re} f, \operatorname{Im} f\colon X \to \mathbb{R}$.
$\overline{f}\colon X \to \mathbb{C}$

$$\begin{aligned} \overline{f}(z) &= \overline{f(z)} \\ &= \operatorname{Re} f(z) - i \operatorname{Im} f(z) \end{aligned}$$

f is continuous iff $\overline{f}$ is continuous

Theorem: (S–W for complex-valued continuous functions)
Let X be a compact metric space and let $\mathcal{A}$ be a subalgebra (scalars from $\mathbb{C}$) of

$$C(X,\mathbb{C}) = \{\, f\colon X \to \mathbb{C} : f \text{ continuous} \,\}$$

which contains all constants (from $\mathbb{C}$), separates points and is closed under conjugation (meaning $f \in \mathcal{A} \implies \overline{f} \in \mathcal{A}$).
Then $\mathcal{A}$ is (uniformly) dense in $C(X,\mathbb{C})$.

Example: $X = \{\, z \in \mathbb{C} : |z| = 1 \,\}$
$\mathcal{A} = \left\{ \sum_{n=-N}^{N} a_n z^n : a_n \in \mathbb{C},\, N \in \mathbb{N} \right\}$ trigonometric polynomials
For $z \in X$, $\overline{z} = z^{-1} = \frac{1}{z}$
If $f^{70)} = \sum_{n=-N}^{N} a_n z^n$, $\overline{f}(z) = \sum \overline{a_n z}^n = \sum_{n=-N}^{N} a_n z^{-n} \in \mathcal{A}$
So $\mathcal{A}$ is an algebra that contains the constants, separates points and is closed under conjugation.
$C(X,\mathbb{C}) \approx C([0,2\pi],\mathbb{C})$ and 2π periodic
$\mathcal{A} = \left\{ \sum_{n=-N}^{N} a_n e^{in\theta} \right\}$

$z = e^{i\theta}$, $\theta \in [0, 2\pi]$
$f(z) = f(e^{i\theta}) = g(\theta)$

figure: unit circle in $\mathbb{C}$

Let $B = \left\{ \sum_{n=0}^{N} a_n z^n : a_n \in \mathbb{C},\, n \in \mathbb{N} \right\}$

- algebra, contains constants, separates points
- B is not dense: $f(z) = \frac{1}{z} \notin$ closure B yet $\frac{1}{z} \in C(X,\mathbb{C})$

Say $f = \lim f_n$, $f_n \in B$
$f(e^{i\theta}) = \lim f_n(e^{i\theta})$ uniformly in θ

$$\int_0^{2\pi} \overline{f} f_n \,\mathrm{d}\theta = \int_0^{2\pi} e^{i\theta} \sum_{k=0}^{N_n} a_k^{(n)} e^{ik\theta} \,\mathrm{d}\theta$$

$\overline{f}(z) = z$

$$= \sum_{k=0}^{N_n} a_k^{(n)} \int_0^{2\pi} e^{i(k+1)\theta} \,\mathrm{d}\theta = 0$$

$$\begin{aligned}
\left| \int_0^{2\pi} \overline{f} f_n - \int_0^{2\pi} \overline{f} f \,\mathrm{d}\theta \right| &= \int_0^{2\pi} |\overline{f}(f_n - f)| \,\mathrm{d}\theta \\
&\le \int_0^{2\pi} |\overline{f}||f_n - f| \,\mathrm{d}\theta \\
&\le M \int_0^{2\pi} |f_n - f| \,\mathrm{d}\theta \\
&< M\epsilon \cdot 2\pi \text{ for } n \text{ sufficiently large}
\end{aligned}$$

$$\begin{aligned}
\implies {}^{71)} \int_0^{2\pi} \overline{f} f_n \,\mathrm{d}\theta &\to \int_0^{2\pi} |f|^2 \,\mathrm{d}\theta \\
&= \int_0^{2\pi} 1 \,\mathrm{d}\theta \\
&= 2\pi
\end{aligned}$$

---

70) $\in \mathcal{A}$

• contradiction

Proof of S–W for complex-valued functions

$$\text{Let } \mathcal{A}_{\mathbb{R}} = \{\text{real-valued functions in } \mathcal{A}\} \subseteq C(X)$$

- contains all real valued constant functions

$\mathcal{A}$-algebra over $\mathbb{R}$
If $f \in \mathcal{A}$ then $\overline{f} \in \mathcal{A} \implies f + \overline{f} = 2 \operatorname{Re} f \in \mathcal{A}$
$\implies \operatorname{Re} f \in \mathcal{A} \implies \operatorname{Re} f \in \mathcal{A}_{\mathbb{R}}$
Similarly $\operatorname{Im} f \in \mathcal{A} \implies \operatorname{Im} f \in \mathcal{A}_{\mathbb{R}}$.

If $x \neq y$ then there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$
$\implies$ At least one of $\operatorname{Re} f(x) \neq \operatorname{Re} f(y)$ or $\operatorname{Im} f(x) \neq \operatorname{Im} f(y)$
Therefore $\mathcal{A}_{\mathbb{R}}$ separates points.

By S–W Theorem, $\mathcal{A}_{\mathbb{R}}$ is dense in $C(X)$
Let $f \in C(X, \mathbb{C})$ and let $\epsilon > 0$.
Then $\operatorname{Re} f$, $\operatorname{Im} f \in C(X)$ so there exist g, $h \in \mathcal{A}_{\mathbb{R}}$ such that $\|\operatorname{Re} f - g\| < \epsilon$ and $\|\operatorname{Im} f - h\| < \epsilon$
Also $g + ih \in \mathcal{A}$: Calculate $\|f - (g + ih)\|$

$$= \|\underbrace{\operatorname{Re} f + i \operatorname{Im} f}_{=f} - (g + ih)\| \leq \|\operatorname{Re} f - g\| + \|i(\operatorname{Im} f - h)\| < 2\epsilon$$

Applications

1. Let $f \in C(X)$, f 1–1
   Then $\left\{ \sum_{n=0}^{N} a_n f^n(x) : a_n \in \mathbb{R},\, n \in \mathbb{N} \right\}$ is dense in $C(X)$
2. Suppose $f \in C[0,1]$ and $\int_0^1 f(x) x^n \, \mathrm{d}x = 0$ for all $n = 0, 1, 2, \ldots$.
   Then $f = 0$.
   Proof: $\int_0^1 f(x) p(x) \, \mathrm{d}x = 0$ for $p(x) =$ polynomial
   Know there exists $p_N \to f$ uniformly for polynomials p_N and so $\int_0^1 \underbrace{f \cdot p_N}_{=0} \, \mathrm{d}x \to \int_0^1 f \cdot f \, \mathrm{d}x = \int_0^1 \|f\|^2 \, \mathrm{d}x$
   $\implies f = 0$.

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Applications of S–W Theorem

(1)
$$\int_0^1 f(x) x^n \, \mathrm{d}x = 0 \qquad \forall n = 0, 1, 2, \ldots$$

$$\implies f = 0$$

Uniqueness Theorem

(2) If f 2π-periodic, continuous function and $\hat{f}(j) = 0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ijx} \, \mathrm{d}x$ $\forall j \in \mathbb{Z}$ then $f \equiv 0$.
Proof: Let $p(x) = \sum_{n=-N}^{N} a_k e^{ikx}$ for any trigonometric polynomials
Then $\frac{1}{2\pi} \int_0^{2\pi} f(x) p(x) \, \mathrm{d}x = 0$
Take $p_N \to \overline{f}$ uniformly.

$$\frac{1}{2\pi} \int_0^{2\pi} f \cdot p_N{}^{72)} \to \frac{1}{2\pi} \int_0^{2\pi} f \cdot \overline{f} = \frac{1}{2\pi} \int_0^{2\pi} |f|^2 \implies f = 0$$

71) $= 0$

(3) $C([0,1]\times[0,1])$

$$\text{Take } \mathcal{A} = \left\{ \sum_{i=1}^{N} f_i(x)g_i(y) : f_i, g_i\colon [0,1]\to\mathbb{R}, \text{ continuous} \right\}$$

- algebra
- contains constants
- separates points

By S–W, $\mathcal{A}$ is dense in $C([0,1]\times[0,1])$

HW (4) $C[a,b]$ is separable, i.e., countable dense set

(5) **Proposition:** Let X be compact and suppose $\mathcal{A}\subseteq C(X)$ is a subalgebra that separates points, but $\overline{\mathcal{A}}\neq C(X)$.
Then there exists $x_0\in X$ such that $f(x_0)=0\ \forall f\in\mathcal{A}$.
Proof: Suppose not. Then $\forall x\in X\ \exists f_x\in\mathcal{A}$ such that $f_x(x)\neq 0$. By multiplying by a suitable scalar, without loss of generality $f_x(x)=2$. By continuity there exists $\delta_x>0$ such that if $y\in B(x,\delta_x)$ then $f_x(y)\geq 1$.
X is compact so take a finite subcover, say

$$B(x_1,\delta_{x_1}),\ldots,B(x_\kappa,\delta_{x_\kappa})$$

$$\text{Put } f(y)=\sum_{i=1}^{\kappa} f_{x_i}^2(y)\in\mathcal{A}$$

If $y\in X$, then there exists i such that $y\in B(x_i,\delta_{x_i})$
$\implies f_{x_i}^2(y)\geq 1$
$\implies f(y)\geq f_{x_i}^2(y)\geq 1 \implies \frac{1}{f}\in C(X)$

$$\text{Consider } \mathcal{A}+\mathbb{R}\equiv\{\, g+\lambda : g\in\mathcal{A},\ \lambda\in\mathbb{R}\,\}\subseteq C(X)$$

$\mathcal{A}+\mathbb{R}$ is an algebra: Take $g_1+\lambda_1$, $g_2+\lambda_2$

$$(g_1+\lambda_1)(g_2+\lambda_2) = \underbrace{g_1g_2+\lambda_2g_1+\lambda_1g_2}_{\in\mathcal{A}} + \underbrace{\lambda_1\lambda_2}_{\in\mathbb{R}}$$

Contains constants because $g=0\in\mathcal{A}$
$\mathcal{A}+\mathbb{R}$ separates points since $\mathcal{A}$ separates points
By S–W Theorem $\mathcal{A}+\mathbb{R}$ is dense in $C(X)$.
So there exists $g_n+\lambda_n\to\frac{1}{f}$ uniformly where $g_n\in\mathcal{A}$, $\lambda_n\in\mathbb{R}$

$$\begin{aligned}|f(y)\cdot g_n(y)+f(y)\lambda_n-1| &= |f(y)|\Big|g_n(y)+\lambda_n-\frac{1}{f(y)}\Big|\\ &\leq \|f\|_\infty\Big|g_n(y)+\lambda_n-\frac{1}{f(y)}\Big|\\ &\to 0 \text{ uniformly}\end{aligned}$$

Hence $\underbrace{fg_n+\lambda_nf}_{\in\mathcal{A}}\to 1$ uniformly
$\implies 1\in\overline{\mathcal{A}}$
So $\overline{\mathcal{A}}$ is a subalgebra of $C(X)$ that contains constants and separates points.
By S–W: $\overline{\mathcal{A}}$ is dense in $C(X)$. But $\overline{\mathcal{A}}$ is closed, therefore $\overline{\mathcal{A}}=C(X)$: contradiction.

Remark: Evaluation map $\phi_{x_0}\colon C(X)\to\mathbb{R}$, $f\mapsto f(x_0)$ [optional]
ϕ_{x_0} linear, multiplicative, continuous onto $\mathbb{R}$

$$\ker\phi_{x_0}=\{\, f : f(x_0)=0\,\}=\phi_{x_0}^{-1}\{0\}$$

[72] $=0$

- closed set
- ideal
- proper ideal

$$C(X)/\ker\phi_{x_0} \cong \mathbb{R} \implies \text{maximal ideal}$$

Theorem: $\{\ker\phi_{x_0} : x_0 \in X\}$: all the maximal ideals in $C(X)$
Previous proposition says $\mathcal{A} \subseteq \ker\phi_{x_0}$
Suppose B algebra with no $x_0 \in X$ such that $f(x_0) = 0\ \forall f \in B$
Apply previous argument to B we see there exists $f \in B$ such that $f(y) \geq 1\ \forall y$
$\implies \frac{1}{f} \in C(X) \implies B$ is not contained in any proper ideal

- Banach algebra.

PMATH 351 Lecture 29: November 23, 2009

Baire Category Theory
Definition: $A \subseteq X$ is called *nowhere dense* if $\operatorname{int}\overline{A} = \emptyset$.
e.g., $\mathbb{Z}$ in $\mathbb{R}$: nowhere dense
$\mathbb{Q}$ in $\mathbb{R}$: fails to be nowhere dense
A is nowhere dense if and only if $\overline{A}$ is nowhere dense
A is called *first category* if $A = \bigcup_{n=1}^{\infty} A_n$ where each A_n is nowhere dense.
e.g., $\mathbb{Q} = \bigcup_{n=1}^{\infty}\{r_n\}$: first category
A is called *second category* otherwise.
If A is nowhere dense then A^{C} is dense.
Why? A set is dense if and only if it intersects every non-empty open set.
Suppose A^{C} is not dense. Then there exists U open, $\neq \emptyset$ such that $U \cap A^{\mathrm{C}} = \emptyset$
$\implies U \subseteq A \implies \operatorname{int}\overline{A} \neq \emptyset$: contradiction.
Proposition: A closed and nowhere dense $\iff A^{\mathrm{C}}$ is open and dense
Proof: $\implies$: ✓
$\impliedby$: Suppose $\operatorname{int}\overline{A}$[73] $= \emptyset$. Hence $\operatorname{int} A \cap A^{\mathrm{C}} = \emptyset$: contradicts A^{C} dense.
Proposition: X is second category if and only if the intersection of every countable family of dense open sets in X is non-empty.
Proof: ($\implies$) Let G_j, $j = 1, 2, \ldots$ be open and dense.
Then G_j^{C} are closed and nowhere dense.

Since X is 2nd category $X \neq \bigcup_1^{\infty} G_j^{\mathrm{C}} \implies \underbrace{\left(\bigcup_1^{\infty} G_j^{\mathrm{C}}\right)^{\mathrm{C}}}_{=\bigcap_{j=1}^{\infty} G_j} \neq \emptyset.$

($\impliedby$) Suppose X is not 2nd category.
Then $X = \bigcup_1^{\infty} \overline{F_j}$ where F_j are closed and nowhere dense.

$$\left(\bigcup_1^{\infty} F_j\right)^{\mathrm{C}} = \emptyset = \bigcap_{j=1}^{\infty} \underbrace{F_j^{\mathrm{C}}}_{\text{open \& dense}}$$

Baire Category Theorem
A complete metric space is second category.
Proof: Let $\{A_n\}_{n=1}^{\infty}$ be open and dense
Show $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$
Let $x_1 \in A_1$ and let U_1 be an open ball[74] containing x_1, $U_1 \subseteq A_1$.
A_2 is dense so there exists $x_2 \in \underbrace{A_2 \cap U_1}_{\text{open}}$.

---
[73] $= A$
[74] $= B(x_1, r_1)$

Since $A_2 \cap U_1$ is open there exists an open set $U_2 \ni x_2$, $U_2 \subseteq A_2 \cap U_1$[75] and $\operatorname{diam} U_2 \leq \frac{1}{2} \operatorname{diam} U_1$ and $\overline{U}_2 \subseteq U_1$

$$(B(x_2, r) \subseteq B(x_1, r_1) \implies \overline{B(x_2, \frac{r}{2})} \subseteq B(x_2, r) \subseteq B(x_1, r_1))$$

Proceed inductively to get open sets $U_n \ni x_n$, $U_n \subseteq \bigcap_1^n A_j$, $\overline{U_n} \subseteq U_{n-1}$, $\operatorname{diam} U_n \leq \frac{1}{2} \operatorname{diam} U_{n-1}$ (so $\operatorname{diam} U_n \to 0$)
Claim $\{x_n\}_1^\infty$ is a Cauchy sequence.
Let $\epsilon > 0$. Pick N such that $\operatorname{diam} U_N < \epsilon$.
If $n, m \geq N$ then $x_n, x_m \in U_N$ (as U_js are nested)
$\implies d(x_n, x_m) \leq \operatorname{diam} U_N < \epsilon$.
Since the space is complete, $x_n \to x$.
Notice $x_n \in \overline{U}_N$ for all $n \geq N \implies x \in \overline{U}_N \subseteq U_{N-1} \subseteq \bigcap_1^{N-1} A_j$
This is true for all $N \implies x \in \bigcap_1^\infty A_j \implies \bigcap_1^\infty A_j \neq \emptyset \implies X$ is second category.
Corollary: $\mathbb{R}$ is uncountable
Proof: $\mathbb{R}$ is second category.
Corollary: A non-empty perfect set E in a complete metric space is uncountable.
Proof: Say $E = \bigcup_{n=1}^\infty \{r_n\}$. E being a closed subset of a complete metric space is complete. Therefore E is second category. This implies $\{r_n\}$ is open for some n.
So there exists $\epsilon > 0$ such that $B(r_n, \epsilon) = \{r_n\}$
But r_n is an accumulation point of $E \implies B(r_n, \epsilon) \cap B(E \setminus \{r_n\}) \neq \emptyset$

- contradiction

Proposition: The set E of functions in $C[0,1]$ which have a derivative at (even) one point of $(0,1)$ is first category.
Corollary: The set of nowhere differentiable continuous functions is second category.
Proof: (exercise) Union of two first category sets is first category.
Proof of proposition:

$$\text{Put } E_n = \left\{ f \in C[0,1] : \exists x \in [0, 1 - \tfrac{1}{n}] \text{ such that } \forall h \in (0, \tfrac{1}{n}],\ \frac{|f(x+h) - f(x)|}{h} \leq n \right\}.$$

If f is differentiable at $x_0 \in (0,1)$ then there exists n_1 such that $x_0 \in [0, 1 - \frac{1}{n_1}]$ and there exists n_2 such that if $0 < h \leq \frac{1}{n_2}$ then

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq |f'(x_0)| + 1$$
$$\leq n_3$$

Take $n = \max(n_1, n_2, n_3) \implies f \in E_n$
Shown $E \subseteq \bigcup_{n=1}^\infty E_n$

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Proposition: The set of functions $E \subseteq C[0,1]$ which have a derivative at one point of $(0,1)$ is first category.
Proof:

$$\text{Put } E_n = \left\{ f \in C[0,1] : \exists x \in [0, 1 - 1/n] \text{ such that } \forall h \in (0, 1/n],\ \frac{|f(x+h) - f(x)|}{h} \leq n \right\}$$

Show

(1) $E \subseteq \bigcup_{n=1}^\infty E_n$

(2) E_n closed

[75] $\subseteq A_2 \cap A_1$

(3) E_n have empty intersection

$$\text{Then } E \overset{(1)}{=} \bigcup_{n=1}^{\infty} (E_n \cap E)$$
$$\overline{E_n \cap E} \subseteq \overline{E_n} \overset{(2)}{=} E_n$$
$$\text{int}(\overline{E_n \cap E}) \subseteq \text{int}\, E_n \overset{(3)}{=} \emptyset$$

$\implies$ $E_n \cap E$ are nowhere dense
E is first category
Step 1: Let $f \in E$, say $f'(x_0)$ exists for $x_0 \in (0,1)$
Then there exists n_1 such that $x \in [0, 1 - 1/n_1]$
There exists n_2 such that $|h| < 1/n_2$ then $\left|\frac{f(x_0+h)-f(x_0)}{h} - f'(x_0)\right| \le 1$

$$\implies \frac{|f(x_0+h)-f(x_0)|}{h} \le 1 + f'(x_0) \quad \forall\, 0 < h \le 1/n_2$$
$$\le n_3$$

Put $n = \max(n_1, n_2, n_3) \implies f \in E_n$
$\implies E \subseteq \bigcup_{n=1}^{\infty} E_n$

(3) Let $f \in E_n$ and let $\epsilon > 0$
Show there exists $g \in C[0,1]$ such that $g \in B(f, \epsilon)$, i.e., $\|g - f\| < \epsilon$, but $g \notin E_n$.
i.e., for all $x \in [0, 1-1/n]$, there exists $h \in (0, 1/n]$ such that

$$\left|\frac{g(x+h)-g(x)}{h}\right| > n$$

Get polynomial P such that $\|f - P\| < \epsilon/2$ (by S–W)
Let $M > \sup_{x\in[0,1]}|P'(x)|$ (can do as $P' \in C[0,1]$)
Let Q be continuous piecewise linear with slope $\pm(M+n+1)$ and $0 \le Q \le \epsilon/2$ — figure: periodic sawtooth between 0 and 1; peak of $\epsilon/2$
Put $g = P + Q \in C[0,1]$

$$\|g - f\| = \|P + Q - f\| \le \|P - f\| + \|Q\|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon$$

$$\frac{|g(x+h)-g(x)|}{h} = \frac{|P(x+h)-P(x)+Q(x+h)-Q(x)|}{h}$$
$$\ge \frac{|Q(x+h)-Q(x)|}{h} - \frac{|P(x+h)-P(x)|}{h}$$
$$\ge M + n + 1 - M \qquad \text{(for small } h)$$
$$= n + 1 > n$$

$\implies g \notin E_n$

(2) Prove E_n is closed.
Suppose $f_m \in E_n$ and $f_m \to f$ (uniformly)
Need to prove $f \in E_n$.
For each m, there exists $x_m \in [0, 1-1/n]$ such that for all $h \in (0, 1/n]$

$$\frac{|f_m(x_m+h)-f_m(x_m)|}{h} \le n \tag{3}$$

By B–W there exists $x_{m_j} \to x_0 \in [0, 1-1/n]$
By relabeling, if necessary, (and throwing away functions not in the subsequent f_{m_j}) we can

assume $x_m \to x_0$.
Fix $h \in (0, 1/n]$. Fix $\epsilon > 0$.
Pick M_1 such that $\|f_m - f\| < \frac{\epsilon h}{4}$ for all $m \geq M_1$ (2)
f is uniformly continuous. There exists $\delta > 0$ such that $|x - y| < \delta$
$\implies |f(x) - f(y)| < \frac{\epsilon h}{4}$ (1)
Pick M_2 such that $|x_m - x_0| < \delta$ if $m \geq M_2$ and then let $M = \max(M_1, M_2)$

$$\begin{aligned}\frac{|f(x_0+h)-f(x_0)|}{h} &\leq \frac{|f(x_0+h)-f(x_M+h)|}{h} + \frac{|f(x_M+h)-f_M(x_M+h)|}{h} \\ &\quad + \frac{|f_M(x_M+h)-f_M(x_M)|}{h} + \frac{|f_M(x_M)-f(x_M)|}{h} + \frac{|f(x_M)-f(x_0)|}{h} \\ &< \frac{\epsilon h/4}{h}^{76)} + \frac{\|f-f_M\|}{h}^{77)} + n^{78)} + \|f_M - f\|^{79)} + \frac{\epsilon \not{h}/4}{\not{h}}^{80)} \\ &< \epsilon/4 + \frac{\epsilon \not{h}/4}{\not{h}} + n + \epsilon/4 + \epsilon/4 \\ &= n + \epsilon\end{aligned}$$

$|x_0 + h - (x_M + h)| = |x_0 - x_M| < \delta$

True for all $\epsilon > 0$, therefore $\frac{|f(x_0+h)-f(x_0)|}{h} \leq n$ for all $h \in (0, 1/n]$
$\implies f \in E_n$. Therefore E_n is closed.

Banach Contraction Mapping Principle
Let X be a complete metric space and let $T\colon X \to X$ be a contraction i.e., exists $r < 1$ such that $d(T(x), T(y)) \leq rd(x, y)$ for all $x, y \in X$
Then T is continuous and has a unique fixed point i.e., point x such that $T(x) = x$.

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Banach Contraction Mapping Principle
$T\colon X \to X$ is a contraction if there exists $r < 1$ such that $d(T(x), T(y)) \leq rd(x, y)$ for all $x, y \in X$

Theorem: If X is a complete metric space and $T\colon X \to X$ is a contraction, then T is a continuous map and has a unique fixed point, i.e., there exists $x \in X$ such that $T(x) = x$.
Proof: In fact a contraction is uniformly continuous.
Given $\epsilon > 0$ take $\delta = \epsilon/r$ and then $d(x, y) < \delta$
$\implies d(T(x), T(y)) \leq r \cdot d = \epsilon$

Take $x_0 \in X$. Look at $T(x_0)$, $T(T(x_0)) = T^2(x_0)$
...
Let $x_1 = T(x_0)$, $x_{n+1} = T(x_n) = T^2(x_{n-1}) = \cdots = T^{n+1}(x_0)$
First check $\{x_n\}_1^\infty$ is a Cauchy sequence.
Start by looking at $d(x_n, x_{n+1}) = d(T(x_{n-1}), T(x_n))$

$$\leq rd(x_{n-1}, x_n) = rd(T(x_{n-2}), T(x_{n-1})) \leq r^2 d(x_{n-2}, x_{n-1}) = \cdots = r^n d(x_0, x_1)$$

Assume $m > n$. Say $m = n + k$.

$$\begin{aligned}d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-1}, x_{n+k}) \\ &\leq r^n d(x_0, x_1) + r^{n+1} d(x_0, x_1) + \cdots + r^{n+k-1} d(x_0, x_1) \\ &= d(x_0, x_1)(r^n + r^{n+1} + \cdots + r^{n+k-1}) \\ &\leq d(x_0, x_1) \sum_n^\infty r^j \to 0 \text{ as } n \to \infty\end{aligned}$$

76) by (1)
77) (2)
78) by (3)
79) (2)
80) by (1)

Hence $\{x_n\}$ is Cauchy
As X is complete there exists $y \in X$ such that $x_n \to y$

$$\text{By continuity } T(x_n) \to T(y)$$
$$\parallel$$
$$x_{n+1} \to y$$

Therefore $T(y) = y$. So y is a fixed point of T.
Suppose z was also a fixed point of T

$$d(z, y) = d(T(z), T(y)) \leq r d(z, y)$$

Since $r < 1 \implies d(z, y) = 0$, i.e., $z = y$

Application to Solving an Integral Equation
Suppose $k(x, y)\colon [0, 1] \times [0, 1] \to \mathbb{R}$, continuous
Consider the equation

$$f(x) = A + \int_0^x k(x, y) f(y)\, \mathrm{d}y. \qquad (*)$$

Find continuous f which satisfies this.
e.g., $k = 1,\ A = 1,\ f(x) = 1 + \int_0^x f(y)\, \mathrm{d}y$

$$g(x) = \int_0^x f(y)\, \mathrm{d}y \text{ is differentiable} \implies f \text{ is differentiable}$$

$g'(x) = f(x)$ by Fundamental Theorem of Calculus
$\implies f'(x) = 0 + f(x) \implies f(x) = ce^x$
Furthermore $f(0) = 1 + \int_0^0 f(y) = 1 \implies c = 1,\ f(x) = e^x$

Theorem: If $\sup_{x \in [0,1]} \int_0^1 |k(x, y)|\, \mathrm{d}y = \lambda < 1$ then $(*)$ has a unique solution.
Proof: Define $T\colon C[0, 1] \to C[0, 1]$ by $T(f)(x) = A + \int_0^x k(x, y) f(y)\, \mathrm{d}y$.
We want a fixed point for T.
Verify $T(f)(x) \in C[0, 1]$. figure: $0 < z < x$
Without loss of generality $x > z$

$$\begin{aligned}
|Tf(x) - Tf(z)| &= \left| \int_0^x k(x, y) f(y)\, \mathrm{d}y - \int_0^z k(z, y) f(y)\, \mathrm{d}y \right| \\
&\leq \left| \int_0^z (k(x, y) - k(z, y)) f(y)\, \mathrm{d}y \right| + \left| \int_z^x k(x, y) f(y)\, \mathrm{d}y \right| \\
&\leq \int_0^z \underbrace{|k(x, y) - k(z, y)|}_{(1)} |f(y)|\, \mathrm{d}y + \int_z^x \underbrace{|k(x, y)|}_{(2)} |f(y)|\, \mathrm{d}y
\end{aligned}$$

k is uniformly continuous. Given $\epsilon > 0$ get δ, i.e., $\|(x, y) - (z, y)\| < \delta \implies |k(x, y) - k(z, y)| < \epsilon$.
f is bounded, say $\|f\| < M$.
Let $|x - z| < \sout{\min(\delta, \epsilon)}$ δ.
Then $\|(x, y) - (z, y)\| = |x - z| < \delta$
$\implies |k(x, y) - k(z, y)| < \epsilon$.
$\implies (1) \leq \int_0^z \epsilon \cdot M\, \mathrm{d}y = z\epsilon M \leq \epsilon M$
(2): Also $\|k\| \leq M' \implies (2) \leq \int_z^x M'M\, \mathrm{d}y = |x - z| M'M < \delta M'M \leq \epsilon M'M$.

$$|Tf(x) - Tf(z)| \leq (1) + (2) \leq \epsilon M + \epsilon M'M = \epsilon(\text{constant})$$

$\implies Tf(x)$ is continuous
$C[0, 1]$ is a complete metric space.

Verify T is a contraction.

$$\begin{aligned} d(Tf, Tg) &= \|Tf - Tg\| \\ &= \sup_{x\in[0,1]} |Tf(x) - Tg(x)| \end{aligned}$$

$$\begin{aligned} |Tf(x) - Tg(x)| &= \left| \int_0^x k(x,y)f(y)\,\mathrm{d}y - \int_0^x k(x,y)g(y)\,\mathrm{d}y \right| \\ &\le \left| \int_0^x k(x,y)(f(y)-g(y))\,\mathrm{d}y \right| \\ &\le \int_0^x |k(x,y)||f(y)-g(y)|\,\mathrm{d}y \\ &\le \|f-g\| \int_0^1 |k(x,y)|\,\mathrm{d}y \\ &\le \lambda^{81)}\|f-g\| = \lambda d(f,g) \end{aligned}$$

Therefore $\|Tf - Tg\| \le \lambda\|f-g\|$
Thus T is a contraction and therefore the integral equation has a unique solution in $C[0,1]$ by Banach Contraction Mapping Principle.

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Example: $T\colon [1,\infty) \to [1,\infty)$

$$\begin{aligned} T(x) &= x + 1/x \\ |T(x) - T(y)| &= |x - y - \tfrac{1}{y} + \tfrac{1}{x}| \\ &= |x - y - \tfrac{x-y}{xy}| \\ &= |x-y||1 - \tfrac{1}{xy}| \\ &< |x-y| \end{aligned}$$

But $T(x) \ne x$ so no fixed point.

Picard's Theorem
Terminology: Say $\Phi\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ is *Lipschitz in y-variable* if there exists a constant L such that

$$|\Phi(x,y) - \Phi(x,z)| \le L|y-z| \qquad \forall x\in[a,b] \ \& \ \forall y,z\in\mathbb{R}$$

Global Picard Theorem
Suppose $\Phi\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ is continuous and Lipschitz in y-variable. Then the differential equation

$$F'(x) = \Phi(x, F(x)), \quad F(a) = c$$

has a unique solution.
Proof: Define $T\colon C[a,b]\to C[a,b]$

$$\text{by } TF(x) = c + \int_a^x \Phi(t, F(t))\,\mathrm{d}t.$$

If $F\in C[a,b]$ then $G(t) = \Phi(t, F(t))$ is continuous.
By the Fundamental Theorem of Calculus $TF(x)$ is differentiable, so $TF \in C[a,b]$ as claimed. $(TF)'(x) = \Phi(x, F(x))$ by Fundamental Theorem of Calculus.
Suppose F is a fixed point of T.

$$\begin{aligned} TF(x) &= F(x) \\ F'(x) &= (TF)'(x) = \Phi(x, F(x)) \text{ and } TF(a)^{82)} = F(a) \end{aligned}$$

81) contraction factor

Thus F satisfies the initial value differential equation.
Conversely, if $F'(x) = \Phi(x, F(x))$ and $F(a) = c$ then $(TF)'(x) = F'(x)\ \forall x \in [a, b]$

$$\begin{aligned} &\implies TF(x) = F(x) + \text{constant} \\ &\implies TF(a)^{82)} = F(a)^{82)} + \text{constant} \end{aligned}$$

so constant $= 0 \implies TF(x) = F(x)$ so F is a fixed point of T.
Can't call on BCMP directly, because T might not be a contraction. But we use same method of proof.
Start with $F_0(x) = c$. Put $F_{k+1}(x) = TF_k(x)$.
Let L be the Lipschitz factor of Φ
Let $M = \max_{a \le x \le b} |\Phi(x, c)|$

$$\begin{aligned} |F_1(x) - F_0(x)| &= |Tc(x) - c| \\ &= \left| c + \int_a^x \Phi(t, c)\,\mathrm{d}t - c \right| \\ &\le \int_a^x |\Phi(t, c)|\,\mathrm{d}t \le M(x - a) \end{aligned}$$

Inductively, we assume $|F_k(x) - F_{k-1}(x)| \le \frac{L^{k-1}M(x-a)^k}{k!}\ \forall x \in [a, b]$

$$\begin{aligned} \text{Then } |F_{k+1}(x) - F_k(x)| &= |T(F_k)(x) - T(F_{k-1})(x)| \\ &= \left| c + \int_a^x \Phi(t, F_k(t))\,\mathrm{d}t - \left(c + \int_a^x \Phi(t, F_{k-1}(t))\,\mathrm{d}t \right) \right| \\ &\le \int_a^x |\Phi(t, F_k(t)) - \Phi(t, F_{k-1}(t))|\,\mathrm{d}t \\ &\le \int_a^x L|F_k(t) - F_{k-1}(t)|\,\mathrm{d}t \qquad \text{by Lipschitz property} \\ &\le \int_a^x L\,\frac{L^{k-1}M(t-a)^k}{k!}\,\mathrm{d}t \qquad \text{(by inductive assumption)} \\ &= \frac{L^k M}{k!} \cdot \left. \frac{(t-a)^{k+1}}{k+1} \right|_a^x = \frac{L^k M(x-a)^{k+1}}{(k+1)!} \end{aligned}$$

That completes the inductive step.
Next, verify $\{F_n\}$ is uniformly Cauchy.
Fix $x \in [a, b]$ temporarily.

$$\begin{aligned} |F_n(x) - F_m(x)| &\le |F_n(x) - F_{n+1}(x)| + |F_{n+1}(x) - F_{n+2}(x)| + \cdots + |F_{m-1}(x) - F_m(x)| \\ &\le \frac{L^n M}{(n+1)!}(x-a)^{n+1} + \cdots + \frac{L^{m-1}M}{m!}(x-a)^m \\ &\le \frac{M}{L} \sum_{j=n+1}^{\infty} \frac{(L(x-a))^j}{j!} \le \underbrace{\frac{M}{L} \sum_{j=n+1}^{\infty} \frac{(L(b-a))^j}{j!}}_{\text{Tail of convergent series}^{83)} \text{ so } < \epsilon \text{ if } n \ge N} \end{aligned}$$

Therefore $\{F_n\}$ is a Cauchy sequence in $C[a, b]$ so $F_n \to F$ uniformly.

82) $= c$

83) $\left(\exp(L(b-a)) = \sum_0^\infty \frac{(L(b-a))^j}{j!}\right)$

Need to prove T is a continuous function

$$\begin{aligned}|TF(x) - TG(x)| &\leq \left|\int_a^x |\Phi(t, F(t)) - \Phi(t, G(t))|\, \mathrm{d}t\right| \\ &\leq \int_a^x L|F(t) - G(t)|\, \mathrm{d}t \\ &\leq L\|F - G\| \int_a^x \mathrm{d}t \\ &\leq L(b-a)\|F - G\|\end{aligned}$$

So $\|TF - TG\| \leq L(b-a)\|F - G\|$
$\implies T$ is continuous.
$T(F_n)$[84] $\to T(F)$ by continuity of T
Therefore $TF = F$.
So F solves the initial-value differential equation.
Suppose G is another solution to differential equation.
Then also $TG = G$.

$$\begin{aligned}\|F - G\| &= \|TF - TG\| = \|T^k F - T^k G\| \\ &\leq \|F - G\| \underbrace{\frac{(L(b-a))^k}{k!}}_{\to 0 \text{ as } k \to \infty} \qquad \text{(by similar arguments)} \\ &\implies \|F - G\| = 0 \implies F = G\end{aligned}$$

Actually valid for $\Phi\colon [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$.
Example:

$$y'' + y + \sqrt{y^2 + (y')^2} = 0$$
$$y(0) = a_0, \quad y'(0) = a_1$$

Let $Y = (y_0, y_1)$
Define $\Phi(x, y_0, y_1)$[85] $= (y_1, -y_0 - \sqrt{y_0^2 + y_1^2}) = (y_1, -y_0 - \|Y\|)$

$$Y'^{86)} = \Phi(x, Y) = \left(y_1, -y_0 - \sqrt{y_0^2 + y_1^2}\right)$$

$\implies y_0' = y_1$

$$y_0'' = y_1'' = -y_0 - \sqrt{y_0^2 + y_1^2} = -y_0 - \sqrt{y_0^2 + (y_0')^2}$$
$$y_0'' + y_0 + \sqrt{y_0^2 + (y_0')^2} = 0$$

PMATH 351 Lecture 33: December 2, 2009

Global Picard Theorem
$\Phi\colon [a, b] \times \mathbb{R} \to \mathbb{R}$, continuous and Lipschitz in y variable. Then the differential equation

$$F'(x) = \Phi(x, F(x)), \qquad F(a) = c$$

has a unique solution.
Example: $y'' + y + \sqrt{y^2 + (y')^2} = 0$, $y(0) = a_0$, $y'(0) = a_1$
Let $Y = (y_0, y_1)$, and

$$\Phi(x, Y) = (y_1, -y_0 - \|Y\|) \qquad (*)$$

$Y(0) = (a_0, a_1)$
$Y' = (y_0', y_1')$

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84) $F_{n+1} \to F$
85) $= \Phi(x, Y)$, $\Phi\colon [0, 1] \times \mathbb{R}^2 \to \mathbb{R}^2$
86) $= (y_0', y_1')$

- Saw if $Y = (y_0, y_1)$ solves $(*)$, then y_0 solves the initial differential equation, and $y_1 = y_0'$.

Check if $\varPhi$ is Lipschitz in Y-variable.

$$\begin{aligned}\|\varPhi(x,Y) - \varPhi(x,Z)\| &= \big\|(y_1, -y_0 - \|Y\|) - (z_1, -z_0 - \|Z\|)\big\| \\ &= \big\|(y_1 - z_1, -y_0 + z_0 - \|Y\| + \|Z\|)\big\| \\ &= \big\|(y_1 - z_1, -y_0 + z_0) + (0, -\|Y\| + \|Z\|)\big\| \\ &\le \big\|(y_1 - z_1, -y_0 + z_0)\big\| + \big\|(0, -\|Y\| + \|Z\|)\big\| \\ &= \big\|(y_1 - z_1, y_0 - z_0)\big\| + \big|\|Z\| - \|Y\|\big| \\ &\le \|Y - Z\| + \|Z - Y\| \\ &= 2\|Y - Z\|\end{aligned}$$

So $\varPhi$ is Lipschitz in Y-variable.
By Global Picard Theorem, there exists a unique solution to the differential equation.

Reminder: In proof of Picard Theorem, Lipschitz condition was used here:

$$\|F_{k+1}(x) - F_k(x)\| = \left\|\int_a^x \varPhi(t, F_k(t)) - \varPhi(t, F_{k-1}(t))\,\mathrm{d}t\right\|$$

Local Picard Theorem
Suppose $\varPhi\colon [a,b] \times [c-\epsilon, c+\epsilon] \to \mathbb{R}$ is continuous, and satisfies a Lipschitz condition in $y \in [c-\epsilon, c+\epsilon]$. Then the differential equation

$$F'(x) = \varPhi(x, F(x)), \qquad F(a) = c$$

has a unique solution for $x \in [a, a+h]$, where $a + h = \min(b, a + \frac{\epsilon}{\|\varPhi\|})$.
Proof: Just check that the iterates $F_k(x)$ stay in $[c-\epsilon, c+\epsilon]$, for all $x \in [a, a+h]$, so we can use the Lipschitz property in exactly the same way as in the proof of the global theorem.
Check: $F_0(x) = c \in [c-\epsilon, c+\epsilon]$

$$\begin{aligned}|F_{k+1}(x) - c| &= \left|c + \int_a^x \varPhi(t, F_k(t))\,\mathrm{d}t - c\right| \\ &\le \int_u^x |\varPhi(t, F_k(t))|\,\mathrm{d}t \\ &\le \|\varPhi\| \int_a^x \mathrm{d}t \\ &= \|\varPhi\|(x-a) \\ &\le h\|\varPhi\| \\ &\le \frac{\epsilon}{\|\varPhi\|}\|\varPhi\| \\ \implies F_{k+1}(x) &\in [c-\epsilon, c+\epsilon], \qquad \forall x \in [a, a+h].\end{aligned}$$

Continuation Theorem
Suppose $\varPhi\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz in y-variable on each compact set $[a,b] \times [-N, N]$, for all N, then the differential equation $F'(x) = \varPhi(x, F(x))$, $F(a) = c$
either has a unique solution on $[a,b]$
or there exists $z \in (a,b)$ such that the differential equation has a unique solution on $[a,z)$, and $\lim_{x\to z^-} |F(x)| = +\infty$.
Example: $y' = y^2$, $y(0) = 1$, for $x \in [0,2]$
$\varPhi(x,y) = y^2$: have Lipschitz condition on every compact set
Solution (by separation of variables) is $y = \frac{1}{1-x}$: get blow up at 1.

PMATH 351 Lecture 34: December 4, 2009

Metric Completion
Definition: Let (X, d_X) be a metric space.
By a *completion of* (X, d_X) we mean a complete metric space (Y, d_Y) and a map $T\colon X \to Y$ such that $d_Y(T(x_1), T(x_2)) = d_X(x_1, x_2)$ and $T(X)$ is dense in Y.

e.g.,

(1) $\mathbb{Q} \subseteq \mathbb{R}$ $\qquad T =$ Identity map

(2) If $X \subseteq X_0$ complete metric space
Take $\mathrm{Id}\colon X \to \overline{X}$ to see $\overline{X}$ is completion of X

Theorem: Every metric space (X, d_X) has a completion
Proof: Fix $x_0 \in X$. Define a family of functions

$$f_x\colon X \to \mathbb{R} \text{ by } f_x(z) = d_X(x, z) - d_X(x_0, z), \quad \forall x \in X.$$

e.g., $f_{x_0}(z) = 0\ \forall z \in X$.
Note:

$$\begin{aligned} d(x, y_1) - d(x, y_2) &\le d(x, y_2) + d(y_2, y_1) - d(x, y_2) \\ &= d(y_2, y_1) \\ \implies |d(x, y_1) - d(x, y_2)| &\le d(y_1, y_2) \\ \text{So } |f_x(z_1) - f_x(z_2)| &= |d(x, z_1) - d(x_0, z_1) - d(x, z_2)^{87)} + d(x_0, z_2)^{88)}| \\ &\le |d(x, z_1) - d(x, z_2)| + |d(x_0, z_1) - d(x_0, z_2)| \le 2d(z_1, z_2) \end{aligned}$$

Thus f_x is (uniformly) continuous.

$$\begin{aligned} \text{Look at } |f_{x_1}(y) - f_{x_2}(y)| &= |d(x_1, y) - d(x_2, y)| \\ &\le d(x_1, x_2) \qquad \forall y \in X \\ \implies \|f_{x_1} - f_{x_2}\| &= \sup_{y \in X} |f_{x_1}(y) - f_{x_2}(y)| \le d(x_1, x_2) \\ \text{But } |f_{x_1}(x_2) - f_{x_2}(x_2)| &= |d(x_1, x_2) - d(x_2, x_2)^{89)}| \\ &= d(x_1, x_2) \\ \text{Therefore } \|f_{x_1} - f_{x_2}\| &= d(x_1, x_2) \end{aligned}$$

In particular, $\|f_{x_1}\| = \|f_{x_1} - f_{x_0}{}^{89)}\| = d(x_1, x_0) < \infty$ so f_{x_1} is bounded for any $x_1 \in X$.
i.e., $f_x \in C_b(X) \leftarrow$ complete metric space

$$\begin{aligned} \text{Consider the map } T\colon X &\to C_b(X) \\ x &\mapsto f_x \end{aligned}$$

$$d_{C_b(X)}(T(x_1)^{90)}, T(x_2)^{91)}) = \|f_{x_1} - f_{x_2}\| = d_X(x_1, x_2)$$

Put $Y = \overline{T(X)}$. Y is complete, being a closed subset of a complete metric space. Y is the completion of X.

87) arrow from first term
88) arrow from second term
89) $= 0$
90) $= f_{x_1}$
91) $= f_{x_2}$