The Best Matrix Conjecture

Curtis Bright

February 13, 2019

Abstract

This note describes the *best matrix conjecture* from combinatorial design theory and the latest results that are known on the conjecture. In particular, examples of best matrices in orders r^2+r+1 for all r up to and including 6 are given.

1 Introduction

Best matrices were introduced by Georgiou, Koukouvinos, and Seberry [2001] and further studied by Koukouvinos and Stylianou [2008] and Đoković [2009]. A quadruple of matrices A, B, C, D are known as *best matrices* if they are square matrices of order n with ± 1 entries and satisfy the following axioms:

(1) $A-I$, $B-I$, $C-I$ are skew matrices and D is a symmetric matrix.

(2) A, B, C, D commute pairwise.

(3) $AA^T+BB^T+CC^T+DD^T$ is the scalar matrix $4nI$.

Note that a matrix X is *symmetric* if $X=X^T$, a matrix X is *skew* if $X=-X^T$, and two matrices X,Y *commute* if $XY=YX$. Best matrices can be used to generate skew Hadamard matrices via a construction introduced by Goethals and Seidel [1970]. In particular, if A, B, C, D are best matrices then the Goethals–Seidel array

$$\begin{pmatrix} A & BR & CR & DR \\ -BR & A & -D^TR & C^TR \\ -CR & D^TR & A & -B^TR \\ -DR & -C^TR & B^TR & A \end{pmatrix}$$

gives a skew Hadamard matrix of order $4n$ where R is the exchange matrix (anti-diagonal identity matrix) of order n.

Furthermore, X is *circulant* if its (i, j) entry is the same as its $(i+1, j+1)$ entry for all indices i and j (reducing mod n if necessary). For the purposes of this note we will only consider circulant best matrices. In this case condition (2) is always satisfied.

Georgiou, Koukouvinos, and Seberry [2001] show that if circulant best matrices exist in odd order n then n must be of the form $(m^2+3)/4$ for odd m. In other words, letting $m = 2r+1$ we have that $n = r^2+r+1$ and the possible values for n are

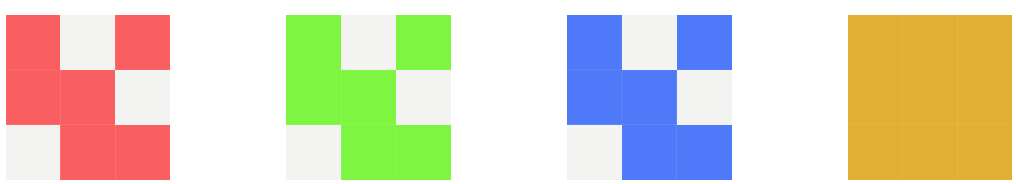
$$\{1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, \dots\}.$$

Georgiou, Koukouvinos, and Seberry [2001] found that best matrices exist for all $r \leq 5$ and for many years no additional best matrices were found. Recently the situation changed as Đoković and Kotsireas [2018] found that best matrices also exist for $r = 6$, i.e., in order $n = 43$.

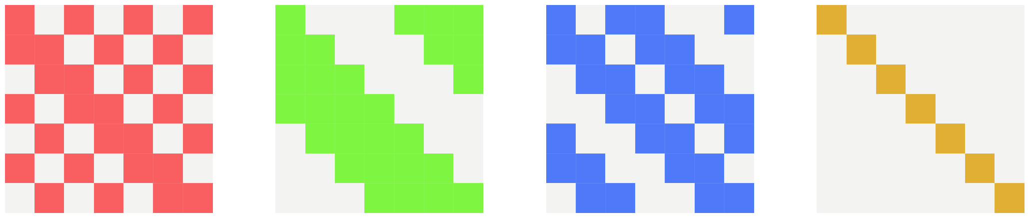
We call the *best matrix conjecture* the conjecture that best matrices exist in all orders of the form r^2+r+1. The conjecture is currently open for each $r \geq 7$.

2 Examples

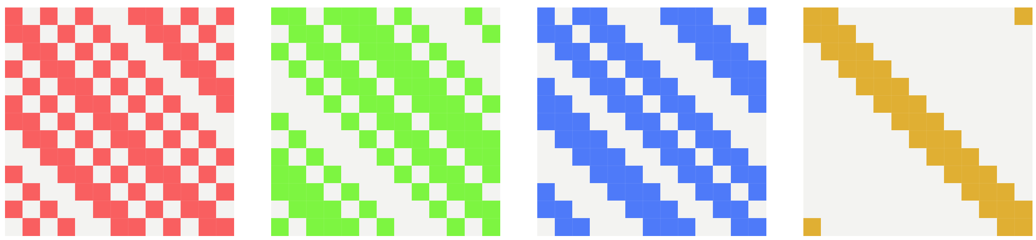
We now explicitly give examples of best matrices for $r = 1, 2, \dots, 6$. The first five examples were found by Georgiou, Koukouvinos, and Seberry [2001] and the sixth was found by Đoković and Kotsireas [2018]. In each example the four matrices A, B, C, D are drawn using a different colour. The coloured squares represent 1 and the grey squares represent -1.



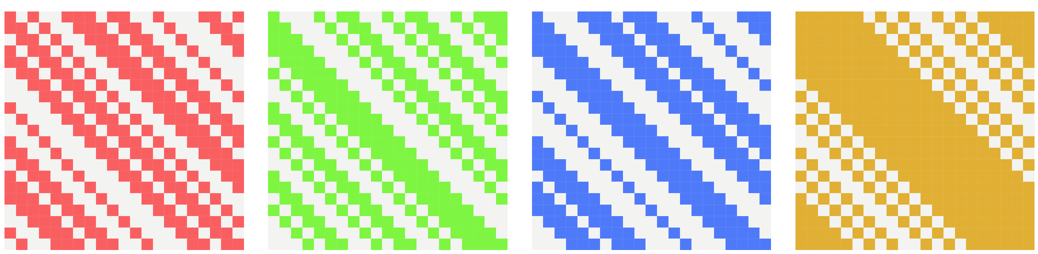
Case $r = 1$: Best matrices of order 3.



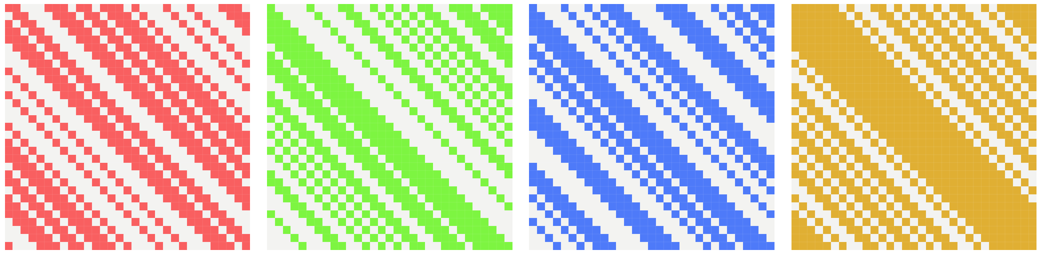
Case $r = 2$: Best matrices of order 7.



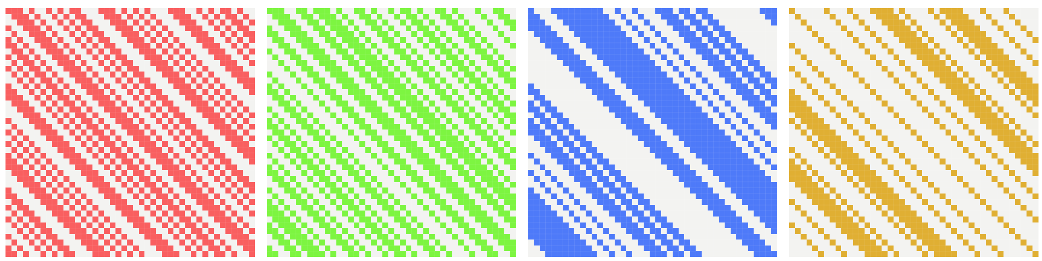
Case $r = 3$: Best matrices of order 13.



Case $r = 4$: Best matrices of order 21.



Case $r = 5$: Best matrices of order 31.



Case $r = 6$: Best matrices of order 43.

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