Vector Rational Number Reconstruction Version 2

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Rational Number Reconstruction

- Given an integer residue $a \in \mathbb{Z}_M$ and a size bound N, the rational number reconstruction problem is to solve

$$da \equiv n \pmod{M}, \qquad d,\, n \leq N$$

for d, $n \in \mathbb{Z}$.

- If $M > 2N^2$ then there is at most one rational number n/d solution.
- For example, consider $a = 25 \in \mathbb{Z}_{97}$ and $N = 6$.

```
> iratrecon(25, 97);
                              3/4
```

- Lo and behold, $4 \cdot 25 \equiv 3 \pmod{97}$.

Vector Rational Number Reconstruction

- Given an integer residue vector $\boldsymbol{a} \in \mathbb{Z}_M^n$ and a size bound N, the vector rational number reconstruction problem is to solve

$$d\boldsymbol{a} \equiv \boldsymbol{n} \pmod{M}, \qquad \|[\, d \mid \boldsymbol{n} \,]\| \leq N$$

for $d \in \mathbb{Z}$ and $\boldsymbol{n} \in \mathbb{Z}^n$.

- For example, consider

$$\boldsymbol{a} = \begin{bmatrix} -23677 & -49539 & 74089 & -21989 & 63531 \end{bmatrix} \in \mathbb{Z}_{195967}^5$$

and $N = 10^4$.

- This has the unique nonzero solution

$$d = 3137 \quad \text{and} \quad \boldsymbol{n} = \begin{bmatrix} -3256 & -2012 & 331 & 891 & -1692 \end{bmatrix},$$

  i.e.,

$$\boldsymbol{a} \equiv \begin{bmatrix} -3256 & -2012 & 331 & 891 & -1692 \end{bmatrix} / \, 3137 \pmod{195967}.$$

- Even though the solution is unique, Maple can't find it because M isn't sufficiently larger than N to ensure entrywise uniqueness.

```
> a := [-23677, -49539, 74089, -21989, 63531]:
> map(iratrecon, a, 195967);
                   -235         211
            [FAIL, ----, FAIL, ---, FAIL]
                   269          303

> map(iratrecon, a, 195967, 3256, 3137);
         2527             -2245              -957
       [----, -2189/4, -----, -1934/9, ----]
          33               37                37
```

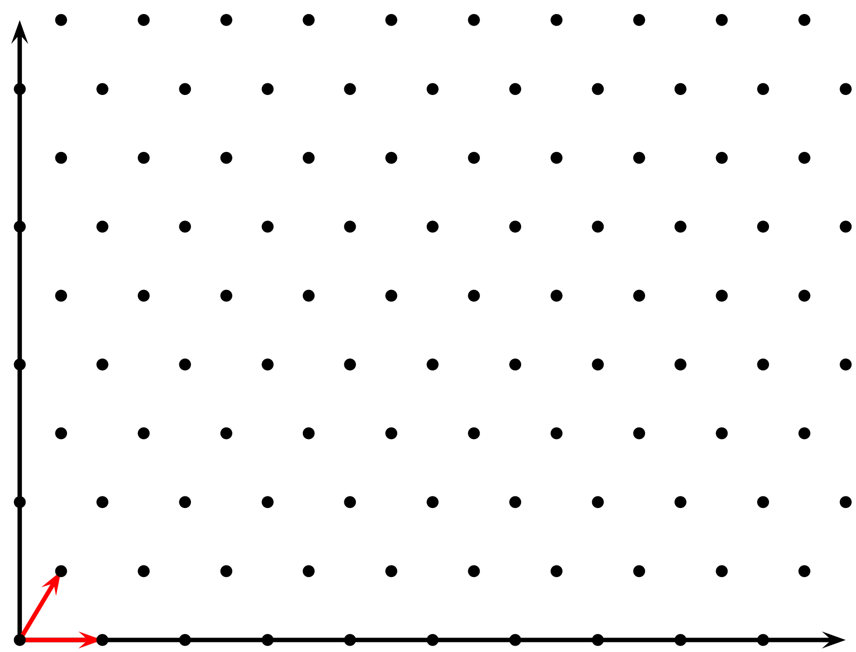
- Finding a common denominator, we see that

$$\boldsymbol{a} \equiv \left[\, -53814 \;\; 16340 \;\; 90815 \;\; -13080 \;\; 12962 \,\right] / \, 14652 \pmod{195967},$$

but this solution vector has norm greater than 10^5, and we wanted one less than 10^4.

Lattices

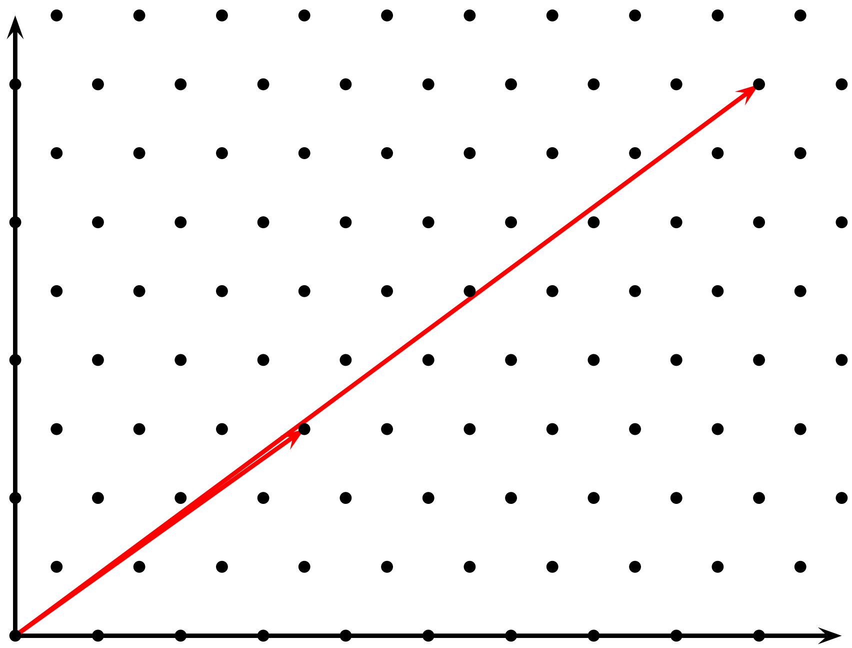
- Given a set of vectors, the lattice generated by them is the set of all integer linear combinations of those vectors:



- A set of linearly independent vectors which generate the same lattice is known a *basis* of the lattice.

Lattice Bases

- Not all bases are created equal, some have needlessly long vectors:



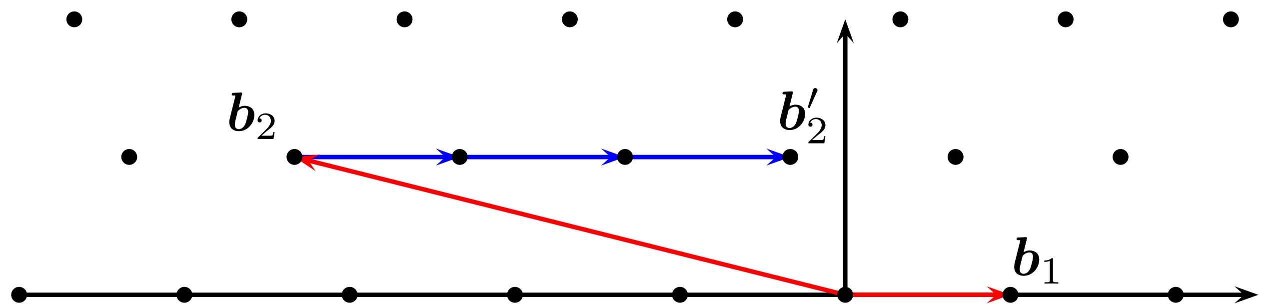
- Many problems, including rational reconstruction, can be posed in the form, ‘given this lattice basis with long vectors, find a short nonzero vector in the lattice’.
- The LLL Algorithm finds a vector within a factor of 2^d of the shortest nonzero vector in a d-dimensional lattice, and it runs in polynomial time in d.

LLL Algorithm

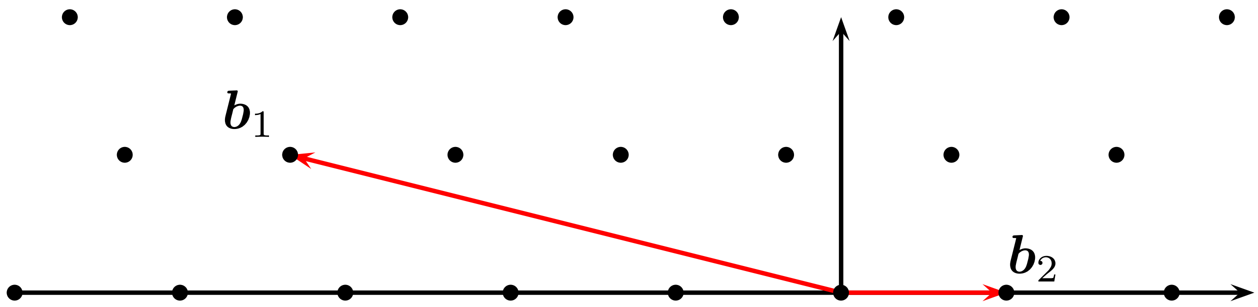
- LLL is based around the concept of *size reduction* of a vector $\boldsymbol{b}_i$ against a set of vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_{i-1}$.
- To do the size reduction against $\boldsymbol{b}_j$, we replace $\boldsymbol{b}_i$ with

$$\boldsymbol{b}_i' \coloneqq \boldsymbol{b}_i - r\boldsymbol{b}_j$$

for the $r \in \mathbb{Z}$ which minimizes $\|\text{proj}_{\boldsymbol{b}_j^*}(\boldsymbol{b}_i')\|$.



- We want short vectors in the set we are size reducing against. If we had instead…



we can't size reduce $\boldsymbol{b}_2$ against $\boldsymbol{b}_1$.

- In a case like this we would want to *swap* $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, and then size-reduce.

- Roughly, the *Lovász condition* is satisfied when $\boldsymbol{b}_i$ and $\boldsymbol{b}_{i-1}$ aren't in a case like this.

LLL Pseudocode

for $i := 2$ to n **do**

  size reduce $\boldsymbol{b}_i$ against $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_{i-1}$

  if Lovász condition not satisfied **then**

    swap $\boldsymbol{b}_{i-1}$ and $\boldsymbol{b}_i$

    $i := \max(i-2, 1)$

- At the conclusion of the loop, the first i vectors are *LLL reduced.*

Rational Reconstruction: Lattice Reformulation

- Consider the lattice generated by the rows of the following $(n+1) \times (n+1)$ integer matrix:

$$\begin{bmatrix} & & & & & M \\ & & & & \iddots & \\ & & & M & & \\ & & M & & & \\ & M & & & & \\ 1 & a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}$$

- For all $d \in \mathbb{Z}$, the vector

$$\begin{bmatrix} d & da_1 & da_2 & da_3 & \cdots & da_n \end{bmatrix}$$

is in this lattice. Because of the first n rows,

$$\begin{bmatrix} d & \mathrm{rem}_M(da_1) & \mathrm{rem}_M(da_2) & \mathrm{rem}_M(da_3) & \cdots & \mathrm{rem}_M(da_n) \end{bmatrix}$$

is also in this lattice.

- Recall we want to solve

$$d\boldsymbol{a} \equiv \boldsymbol{n} \pmod{M}, \qquad \left\| \left[\, d \mid \boldsymbol{n} \,\right] \right\| \le N$$

for d and $\boldsymbol{n}$. Equivalently, we can solve

$$\left\| \left[\, d \mid \mathrm{rem}_M(d\boldsymbol{a}) \,\right] \right\| \le N$$

for d.

- As noted, vectors of this form are in the lattice just considered.
- Therefore, the problem is equivalent to finding vectors shorter than N in the specific lattice we just saw.

Applying LLL Straightforwardly

- The problem instance we saw previously gives raise to the basis matrix

$$\begin{bmatrix} & & & & & 195967 \\ & & & & 195967 & \\ & & & 195967 & & \\ & & 195967 & & & \\ & 195967 & & & & \\ 1 & -23677 & -49539 & 74089 & -21989 & 63531 \end{bmatrix}.$$

- Running LLL on this lattice gives the new basis matrix

$$\begin{bmatrix} -3137 & 3256 & 2012 & -331 & -891 & 1692 \\ \hline -3600 & -8445 & 10430 & -9313 & -10268 & -18111 \\ -4047 & -7044 & 10092 & -8673 & 20465 & -1253 \\ 241 & -23114 & 15088 & 22452 & -8240 & 25545 \\ 28082 & 18517 & 15535 & -14341 & -3081 & -6026 \\ -11836 & 8162 & 10340 & 34921 & 17628 & -27537 \end{bmatrix},$$

from which the first vector gives a solution to our problem.

- In fact, it is not hard to show that the other vectors do not contribute to a vector shorter than N, using the Gram-Schmidt orthogonalization.

Problems with LLL

- Too expensive; running LLL on the previous lattice requires $O(n^6 \log^3 M)$ bit operations.
- LLL approximation factor is 2^n, much too large for large n.

Iterative Reduction

- However, the structure of the lattice permits a kind of iterative reduction.
- For example, consider only reducing the lower-left 2×2 submatrix:

$$\begin{bmatrix} 0 & 195967 \\ 1 & -23677 \end{bmatrix} \xRightarrow{\text{LLL}} \begin{bmatrix} -389 & -96 \\ -149 & 467 \end{bmatrix}$$

- We can use this to help us reduce the lower-left 3×3 submatrix.

- We can tell what the third column would have been, had we kept it around. Note the third column starts out as -49539 times the first column:

$$\left[\begin{array}{cc|c} & 195967 & \\ 1 & -23677 & -49539 \end{array}\right]$$

  and this is always preserved by size reduction and swaps.

- It follows that we have a basis for the lattice generated by the lower-left 3×3 matrix:

$$\begin{bmatrix} & & 195967 \\ & 195967 & \\ 1 & -23677 & -49539 \end{bmatrix} \overset{\text{same lattice}}{\Longleftrightarrow} \begin{bmatrix} & & 195967 \\ -389 & -96 & 19270671 \\ -149 & 467 & 7381311 \end{bmatrix}$$

- We can now run LLL again:

$$\begin{bmatrix} & & 195967 \\ -389 & -96 & 19270671 \\ -149 & 467 & 7381311 \end{bmatrix} \xrightarrow{\text{LLL}} \begin{bmatrix} -538 & 371 & 470 \\ 91 & 1030 & -808 \\ 27089 & 13738 & 20045 \end{bmatrix}$$

- The last vector can now be thrown away, because the last GSO vector has norm larger than N.

Main Contributions

- If $M > 2^{(c+1)/2} N^{1+1/c}$, for $c \in \mathbb{Z}_{>0}$ a small constant which can be chosen, then continuing in this way the row dimension of the matrices will be bounded by $c+1$.
- For $c = O(1)$ the bit complexity is $O(n^2 \log^3 M)$.
- The column dimension of the matrices are bounded by n, but in fact we can get away with *only storing the first column.* This improves the bit complexity to $O(n \log^3 M)$.

- The last step of Dixon's algorithm for linear system solving is to reconstruct a rational vector $\boldsymbol{x} \in \mathbb{Q}^n$ from its modular image $\mathrm{rem}_M(\boldsymbol{x})$ when $M = p^i$.
  - Usual elementwise reconstruction requires $i \approx 2 \log N$.
  - This lattice technique requires $i \approx (1 + \frac{1}{c}) \log N$.