A SAT Solver and Computer Algebra Attack on the Minimum Kochen–Specker Problem (Student Abstract)

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Abstract

The problem of finding the minimum three-dimensional Kochen–Specker (KS) vector system, an important problem in quantum foundations, has remained open for over 55 years. We present a new method to address this problem based on a combination of a Boolean satisfiability (SAT) solver and a computer algebra system (CAS). Our approach improved the lower bound on the size of a KS system from 22 to 24. More importantly, we provide the first computer-verifiable proof certificate of a lower bound to the KS problem with a proof size of 41.6 TiB for order 23. The efficiency is due to the powerful combination of SAT solvers and CAS-based orderly generation.

Introduction

The KS theorem, a fundamental result in Quantum Foundations by Kochen and Specker (1967), rules out non-contextual hidden-variable theories via the existence of a finite set of vectors, referred to as a *KS vector system*. Since 1967, physicists and mathematicians have wondered about the cardinality of the smallest-sized KS vector system, a combinatorial object that witnesses a contradiction between non-contextuality and the SPIN axiom of quantum mechanics (see Table 1). Finding the minimum KS system is not only of scientific and historical interest, but also has applications in quantum information processing.

We present the first implementation of a SAT+CAS tool (a combination of a SAT solver and a Computer Algebra System with proof verification) aimed at solving problems in quantum foundation. We do so by leveraging the SAT+CAS paradigm to incorporate an isomorph-free generation method (as part of our tool, PhysicsCheck[1]) to obtain tighter bounds on the minimum KS problem with orders of magnitude speedup over previous methods. Recently, this paradigm has found wide application in diverse fields that require solving hard combinatorial problems (Bright, Kotsireas, and Ganesh 2022). We implement an extension of the standard Boolean proof certificate format DRAT (Wetzler, Heule, and Hunt Jr 2014) to construct certificates of nonexistence for KS systems.

[1]https://github.com/curtisbright/PhysicsCheck

Authors	Year	Bound
Kochen, Specker	1967	≤ 117
Jost	1976	≤ 109
Conway, Kochen	1990	≤ 31
Arends, Ouaknine, Wampler	2009	≥ 18
Uijlen, Westerbaan	2016	≥ 22
Li, Bright, Ganesh	2022	≥ 23
Li, Bright, Ganesh / Kirchweger, Peitl, Szeider	2023	≥ 24

Table 1: A chronology of the bounds on the size of the minimum KS vector system in three dimensions.

Kochen–Specker Graphs and Systems

A set of 3-dimensional vectors $\mathcal{K}$ has a corresponding **orthogonality graph** $G_{\mathcal{K}} = (V, E)$, where $V = \mathcal{K}$, $E = \{\,(v_1, v_2) : v_1, v_2 \in \mathcal{K}$ and $v_1 \cdot v_2 = 0\,\}$. A graph is **embeddable** if it is a subgraph of an orthogonality graph. It is **010-colorable** if there is a $\{0,1\}$-coloring of the vertices such that no two adjacent vertices are colored 1 and the vertices are not all colored 0 in each triangle. A **KS graph** is an embeddable and non-010-colorable graph, and the minimum KS problem is to find the smallest KS graph. Arends, Ouaknine, and Wampler (2011) proved a number of properties that the smallest KS graph must satisfy. We encode these properties and the non-010-colorability of KS graphs in conjunctive normal form (CNF), solutions of which are referred to as **KS candidates**. If a KS candidate is embeddable, then the corresponding set of vectors is a **KS system**.

Orderly Generation via SAT+CAS

A crucial part of our SAT+CAS tool PhysicsCheck is the combination of a Boolean encoding (the SAT part) with an orderly isomorph-free generation routine (the CAS part). The orderly generation approach was developed independently by Read (1978) and Faradžev (1978). An adjacency matrix M of a graph is **canonical** if every permutation of the graph's vertices produces a matrix lexicographically greater than or equal to M, where the lexicographical order is defined by concatenating the above-diagonal entries of the columns of the adjacency matrix. The orderly generation method is based on the property that if a matrix is not canonical, then all of its extensions are not canonical

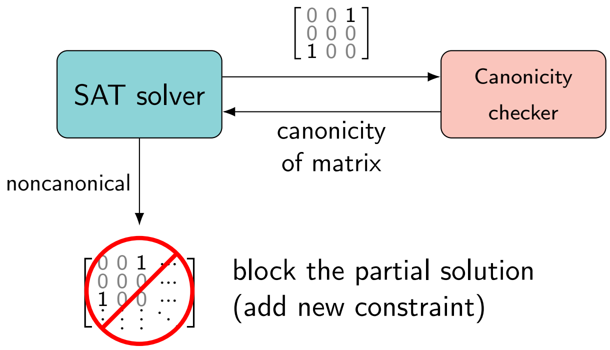

Figure 1: Orderly generation algorithm of SAT+CAS.

and can be discarded. In SAT+CAS, when the SAT solver finds an intermediate/partial matrix, the canonicity of this matrix is determined by a canonicity-checking CAS routine implemented in the PhysicsCheck system. If the matrix is found to be noncanonical, then a "blocking" clause, created via CAS, is dynamically added to the SAT solver, thus removing this matrix and its extensions from the search. Otherwise, the partial matrix may be canonical and the solving continues. As can be seen from Table 2, the SAT+CAS method is orders of magnitude faster than SAT-only or CAS-only approaches. The CAS compared against was the *nauty* graph generator (McKay and Piperno 2014) with the same configuration used by Uijlen and Westerbaan (2016).

Results and Verification

Instances up to order 22 were solved sequentially using the SAT+CAS paradigm on an Intel Xeon E5-2667 CPU. The difficulty of solving order 23 required us to use a parallel cube-and-conquer approach on a cluster of up to 5000 Intel E5-2683 CPUs. All computations are measured in the total CPU time reported by the solver in Table 2. The lower bound was improved to 24 independently by Kirchweger, Peitl, Szeider and Li, Bright, Ganesh in 2023.[2] We estimate that our search in order 21 is about 35,000 times faster than the search by Uijlen and Westerbaan (2016). Furthermore, we achieve comparable runtime to Kirchweger, Peitl, and Szeider (2023) that uses a SAT modulo symmetries (SMS) solver. We found that all KS candidates of order less than 24 are not embeddable, and hence the smallest KS system has at least 24 vectors (Table 1).

The computations were performed by MapleSAT (Liang et al. 2016) combined with a CAS and were verified using a proof produced by the solver in the DRAT format (Wetzler, Heule, and Hunt Jr 2014), except the CAS-derived clauses were prefixed by '**t**' to signify they must be verified separately. The CAS-derived noncanonical blocking clauses are justified via a CAS-derived permutation that provides a witness that the blocked matrix is noncanonical and is safe to block. We have certified the results up to and including order 23. The uncompressed proofs in order 22 and 23 are of size 1.9 and 41.6 TiB. The certifications of orders 22 and 23 were done using a parallel cube-and-conquer solver (Heule et al. 2011) to ensure that each DRAT proof could be verified with at most 4 GiB of memory. In addition to these proofs, we have conducted extensive cross-verification on all the results produced by the SAT solver in PhysicsCheck.

[2]At the 2022 SC-Square workshop, we presented a preliminary version with a lower bound of 23 (Li, Bright, and Ganesh 2022).

order	SAT+CAS	Speedup over SAT	Speedup over CAS
17	0.02 h	8.4×	24.2×
18	0.04 h	123.8×	211.5×
19	0.22 h	883.5×	717.6×
20	1.35 h	timeout	timeout
21	18.12 h	timeout	timeout
22	356.88 h	timeout	timeout
23	52,619.16 h	timeout	timeout

Table 2: Speedup of SAT+CAS over SAT-only and CAS-only. Order 23 was solved with a cube-and-conquer approach.

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