The Quartic Formula Derivation

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Abstract

This article contains an exposition of one possible derivation of the quartic formula. It was originally published in conjunction with the quartic formula poster of Curtis Bright.

Introduction

Consider the arbitrary quartic equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

for real numbers a, b, c, d, e with $a \neq 0$. By the fundamental theorem of algebra this equation has four roots x_1, x_2, x_3, x_4 over the complex numbers. Using the factor theorem gives the factorization

$$ax^4 + bx^3 + cx^2 + dx + e = a(x - x_1)(x - x_2)(x - x_3)(x - x_4).$$

Expanding out the right-hand side gives

$$\begin{gathered} ax^4 - a(x_1 + x_2 + x_3 + x_4)x^3 \\ + a(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)x^2 \\ - a(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)x + ax_1x_2x_3x_4, \end{gathered}$$

and equating coefficients with the original expression gives the following system of equations:

$$\begin{aligned} b &= -a(x_1 + x_2 + x_3 + x_4) \\ c &= a(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) \\ d &= -a(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) \\ e &= ax_1x_2x_3x_4 \end{aligned}$$

At this point, we're basically stuck; this is a complicated non-linear system that we want to solve for x_1, x_2, x_3, x_4.

Some Sleight-of-hand

It turns out that it is now helpful to introduce the variables y_1, y_2, y_3 using the following definitions:

$$\begin{aligned} y_1 &= a(x_1 + x_2 - x_3 - x_4) \\ y_2 &= a(x_1 - x_2 + x_3 - x_4) \\ y_3 &= a(x_1 - x_2 - x_3 + x_4) \end{aligned}$$

And for the real bit of magic, I claim that the following identities hold:

$$\begin{aligned} y_1^2 + y_2^2 + y_3^2 &= 3b^2 - 8ac \\ y_1^2y_2^2 + y_1^2y_3^2 + y_2^2y_3^2 &= 3b^4 + 16a^2c^2 + 16a^2bd - 16ab^2c - 64a^3e \\ y_1y_2y_3 &= -b^3 + 4abc - 8a^2d \end{aligned}$$

The real trick, of course, is where these identities came from—but once someone gives you them, it is not necessary to know how they were derived to check that they are true. It is straightforward, although tedious, to expand out the left sides using the definitions of y_1, y_2, and y_3, and to expand out the right sides using the expressions for b, c, d, and e. The result will be equations in terms of x_1, x_2, x_3, x_4, and a, and one merely needs to check that both sides are identical. [A real magician never reveals their secrets, but if you're curious I've also written a bit about how the identities were computed.]

Finding the y_i

What make the y_i so useful is that one can solve for them using the above identities. It's not immediately obvious how one would go about doing this, since again we have a complicated non-linear system of equations to solve. The trick is to consider a cubic equation which has as its roots y_1^2, y_2^2, and y_3^2; for example, $(y - y_1^2)(y - y_2^2)(y - y_3^2)$. Expanding this out gives

$$y^3 - (y_1^2 + y_2^2 + y_3^2)y^2 + (y_1^2y_2^2 + y_1^2y_3^2 + y_2^2y_3^2)y - (y_1y_2y_3)^2,$$

and the left-hand side of the identities make an appearance! Rewriting this using the right-hand sides gives

$$y^3-(3b^2-8ac)y^2+(3b^4+16a^2c^2+16a^2bd-16ab^2c-64a^3e)y-(-b^3+4abc-8a^2d)^2,$$

and now one can solve this using the cubic formula, and therefore find the roots y_1^2, y_2^2, and y_3^2. Taking square roots gives y_1, y_2, and y_3, except now a problem presents itself: since every number has two square roots, how do you know which one to take?

Actually, the problem isn't as bad as it seems. As long as one ensures that the values for y_i chosen satisfy the third identity $y_1y_2y_3 = -b^3 + 4abc - 8a^2d$, choosing different values for the square roots will just end up causing the x_i to be labelled differently. So you don't have a completely free choice of square roots (in particular, one can't just use principal square roots), but 4 of the 8 possible square root selections will give correct answers.

Finding the x_i

Now that the y_i have been found we are in a great position to use them to find the x_i and thus solve the quartic. Consider the definitions of the y_i augmented with the expression for b:

$$\begin{aligned} b &= -a(x_1 + x_2 + x_3 + x_4) \\ y_1 &= a(x_1 + x_2 - x_3 - x_4) \\ y_2 &= a(x_1 - x_2 + x_3 - x_4) \\ y_3 &= a(x_1 - x_2 - x_3 + x_4) \end{aligned}$$

We know a, b, y_1, y_2, and y_3, and wish to solve this system for x_1, x_2, x_3, and x_4. Since this is a nonsingular linear system, linear algebra allows us to do this! The solution (calculated using Gaussian elimation, for example) is as follows:

$$\begin{aligned} x_1 &= (-b + y_1 + y_2 + y_3)/(4a) \\ x_2 &= (-b + y_1 - y_2 - y_3)/(4a) \\ x_3 &= (-b - y_1 + y_2 - y_3)/(4a) \\ x_4 &= (-b - y_1 - y_2 + y_3)/(4a) \end{aligned}$$

Or more concisely, the solution can be expressed as

$$x = \frac{-b \pm (y_1 \pm y_2) \pm y_3}{4a}$$

where all choices of the $\pm$ signs are chosen with the last two equivalent.

Although this is a perfectly legitimate solution of the quartic, it relies on one "manually" choosing values for square roots so that $\sqrt{y_1^2}\sqrt{y_2^2}\sqrt{y_3^2} = -b^3 + 4abc - 8a^2d$ is satisfied. For simplicity, I've also derived a slight modification which only uses principal square roots.

References

Galois Theory by David A. Cox, in particular see Chapter 12.1C
Galois Theory by Harold M. Edwards, in particular see Section 17
Elements of Abstract Algebra by Allan Clark, in particular see Article 148
Abstract Algebra by Davis S. Dummit and Richard M. Foote, in particular see Chapter 14.7
Galois Theory by Ian Stewart, in particular see Chapter 18.5

The Magic Exposed

The trick is that the y_i were carefully chosen so that $y_1^2+y_2^2+y_3^2$, $y_1^2y_2^2+y_1^2y_3^2+y_2^2y_3^2$, and $y_1y_2y_3$ are symmetric polynomials in the x_is. This means that the value of these polynomials do not change if you permute the x_i. For example, the expression $y_1^2+y_2^2+y_3^2$ is

$$a^2(x_1+x_2-x_3-x_4)^2+a^2(x_1-x_2+x_3-x_4)^2+a^2(x_1-x_2-x_3+x_4)^2$$

and after permuting x_1 and x_2 becomes

$$\begin{aligned}&a^2(x_2+x_1-x_3-x_4)^2+a^2(x_2-x_1+x_3-x_4)^2+a^2(x_2-x_1-x_3+x_4)^2\\=\,&a^2(x_1+x_2-x_3-x_4)^2+a^2(x_1-x_2-x_3+x_4)^2+a^2(x_1-x_2+x_3-x_4)^2\\=\,&y_1^2+y_2^2+y_3^2.\end{aligned}$$

By the fundamental theorem of symmetric polynomials every symmetric polynomial can be written in terms of the elementary symmetric polynomials, which are in fact exactly the expressions derived for b, c, d, and e (up to a factor of $\pm a$). Therefore by this theorem there is some expression for $y_1^2+y_2^2+y_3^2$ in terms of a, b, c, d, and e only. There are various ways of computing what that expression actually is; for example, the three identities I gave were computed with the following Maple code using Gröbner bases:

```
T := lexdeg([y1,y2,y3], [x1,x2,x3,x4], [b,c,d,e]):
G := Groebner[Basis]([
        y1-a*(x1+x2-x3-x4),
        y2-a*(x1-x2+x3-x4),
        y3-a*(x1-x2-x3+x4),
        b-(-a*(x1+x2+x3+x4)),
        c-(a*(x1*x2+x1*x3+x1*x4+x2*x3+x2*x4+x3*x4)),
        d-(-a*(x1*x2*x3+x1*x2*x4+x1*x3*x4+x2*x3*x4)),
        e-(a*(x1*x2*x3*x4))],
     T):
Groebner[Reduce](y1^2+y2^2+y3^2, G, T);
Groebner[Reduce](y1^2*y2^2+y1^2*y3^2+y2^2*y3^2, G, T);
Groebner[Reduce](y1*y2*y3, G, T);
```