Symbolic Sets for Proving Bounds on Rado Numbers

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Abstract

Given a linear equation $\mathcal{E}$ of the form $ax + by = cz$ where a, b, c are positive integers, the k-colour *Rado number* $R_k(\mathcal{E})$ is the smallest positive integer n, if it exists, such that every k-colouring of the positive integers $\{1, 2, \dots, n\}$ contains a monochromatic solution to $\mathcal{E}$. In this paper, we consider $k = 3$ and the linear equations $ax + by = bz$ and $ax + ay = bz$. Using SAT solvers, we compute a number of previously unknown Rado numbers corresponding to these equations. We prove new general bounds on Rado numbers inspired by the satisfying assignments discovered by the SAT solver. Our proofs require extensive case-based analyses that are difficult to check for correctness by hand, so we automate checking the correctness of our proofs via an approach which makes use of a new tool we developed with support for operations on symbolically-defined sets—e.g., unions or intersections of sets of the form $\{f(1), f(2), \dots, f(a)\}$ where a is a *symbolic* variable and f is a function possibly dependent on a. No computer algebra system that we are aware of currently has sufficiently capable support for symbolic sets, leading us to develop a tool supporting symbolic sets using the Python symbolic computation library `SymPy` coupled with the Satisfiability Modulo Theories solver `Z3`.

Keywords

Rado numbers, Ramsey Theory on the Integers, Satisfiability, SMT, Symbolic Computation

1. Introduction

The study of Rado numbers [1] and colouring problems lies at the heart of Ramsey theory and additive combinatorics. In this paper, we present a number of new values and bounds on Rado numbers, as well as an automated framework for proving lower bounds on Rado numbers using symbolic partitioning and satisfiability checking.

1.1. Combinatorial context

For integers a, b, let $[a, b]$ denote the set of integers $\{x : a \leq x \leq b\}$. For a positive integer $k \geq 2$, a system of linear equations is *k-regular* if there exists a positive integer n such that for every k-colouring of $[1, n]$, there exists a monochromatic solution to the system. A system is *regular* if it is k-regular for every integer $k \geq 2$. The following theorem by Rado [1] gives necessary and sufficient conditions for when a linear equation is regular.

Theorem 1.1 (Rado [1]). *For nonzero integers $a_1, a_2, \dots, a_m$ and integer c, let $s = \sum_{i=1}^{m} a_i$. Then:*

- *The equation $\sum_{i=1}^{m} a_i x_i = 0$ is regular if and only if there exists a nonempty set $D \subseteq [1, m]$ such that $\sum_{d \in D} a_d = 0$.*
- *The equation $\sum_{i=1}^{m} a_i x_i = c$ is regular if and only if one of the following two conditions holds: (a) $c/s \in \mathbb{Z}^+$; (b) $c/s \in \mathbb{Z}^-$ and $\sum_{i=1}^{m} a_i x_i = 0$ is regular.*

For nonzero integers $a_1, a_2, \dots, a_m$ and integer c, let the linear equation $\sum_{i=1}^{m-1} a_i x_i + c = a_m x_m$ be represented by $\mathcal{E}(m, c; a_1, a_2, \dots, a_m)$. For a linear equation $\mathcal{E}(m, c; a_1, a_2, \dots, a_m)$ and a positive integer k, the k-colour *Rado number* $R_k(\mathcal{E}(m, c; a_1, a_2, \dots, a_m))$ is defined as the smallest positive integer n, if it exists, from the definition of k-regularity demonstrating that the equation $\mathcal{E}(m, c; a_1, a_2, \dots, a_m)$ is k-regular. Otherwise, we say that $R_k(\mathcal{E}(m, c; a_1, a_2, \dots, a_m))$ is infinite if there is a k-colouring of

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the positive integers with no monochromatic solution to $\mathcal{E}(m, c; a_1, a_2, \ldots, a_m)$. Surprisingly, not much is known about the properties of Rado numbers for $k = 3$. In 1995, Schaal [2] proved that the 3-colour Rado numbers $R_3(\mathcal{E}(3, c; 1, 1, 1))$ are always finite and $R_3(\mathcal{E}(3, c; 1, 1, 1)) = 13c + 14$ for $c \geq 0$. In 2015, Adhikari et al. [3] proved exact formulas for $R_3(\mathcal{E}(4, c; 1, 1, 1, 1))$ and $R_3(\mathcal{E}(5, c; 1, 1, 1, 1, 1))$ with $c \geq 0$. In 2022, Chang, De Loera, and Wesley [4] proved

- $R_3(\mathcal{E}(3, 0; 1, -1, a - 2)) = a^3 - a^2 - a - 1$ for $a \geq 3$ (previously conjectured by Myers [5]),
- $R_3(\mathcal{E}(3, 0, a, -a, (a - 1))) = a^3 + (a - 1)^2$ for $a \geq 3$, and
- $R_3(\mathcal{E}(3, 0; a, -a, b)) = a^3$ for $b \geq 1$, $a \geq b + 2$, and $\gcd(a, b) = 1$.

They also determined exact values of the Rado numbers

- $R_3(\mathcal{E}(3, 0; a, -a, b))$ for $1 \leq a, b \leq 15$,
- $R_3(\mathcal{E}(3, 0; a, a, b))$ for $1 \leq a, b \leq 10$,
- $R_3(\mathcal{E}(3, 0; a, b, c))$ for $1 \leq a, b, c \leq 6$,

as well as the following theorem.

Theorem 1.2. *$R_3(\mathcal{E}(3, 0; 1, 1, a)) = \infty$ for $a \geq 4$ and $R_3(\mathcal{E}(3, 0; a, a, 1)) = \infty$ for $a \geq 2$.*

1.2. Our contributions

In this paper, we have investigated values and bounds of Rado numbers for the equations $\mathcal{E}(3, 0; a, a, b)$ and $\mathcal{E}(3, 0; a, b, b)$. We have computed a number of previously unknown exact values using SAT solvers, extending the results presented by Chang, De Loera, and Wesley [4] to the values of

- $R_3(\mathcal{E}(3, 0; a, b, b))$ for $1 \leq b \leq 15$ and $1 \leq a \leq 30$, and
- $R_3(\mathcal{E}(3, 0; a, a, b))$ for $1 \leq a \leq 15$ and $1 \leq b \leq 25$.

Note that we may assume that a and b are coprime, since if a and b share a common factor g we may divide the coefficients of the equation $\mathcal{E}$ by g without changing its solutions, e.g., $R_3(\mathcal{E}(3, 0; a, b, b)) = R_3(\mathcal{E}(3, 0; a/g, b/g, b/g))$. Guided by the satisfying assignments generated by the SAT solver in the process of computing these values, we proved the following two theorems.

Theorem 1.3. *For coprime positive integers a, b with $a > b \geq 3$ and $a^2 + a + b > b^2 + ba$,*

$$R_3(\mathcal{E}(3, 0; a, b, b)) \geq a^3 + a^2 + (2b + 1)a + 1.$$

Theorem 1.4. *For odd integers $a \geq 7$,*

$$R_3(\mathcal{E}(3, 0; a, a, a + 1)) \geq a^3(a + 1).$$

Based on the computational data we generated and the above lower bounds, we propose the following conjectures.

Conjecture 1.1. *For coprime positive integers a, b with $a > b \geq 3$ and $a^2 + a + b > b^2 + ba$,*

$$R_3(\mathcal{E}(3, 0; a, b, b)) = a^3 + a^2 + (2b + 1)a + 1.$$

Conjecture 1.2. *For coprime positive integers a, b, with a odd, if $3 \leq a < b \leq 2a - 1$ then*

$$R_3(\mathcal{E}(3, 0; a, a, b)) = a^3 b.$$

The proofs of Theorems 1.3 and 1.4 are automatically verified using a symbolic analyzer we developed in Python called AutoCase.[1] The motivation has been to avoid errors in the intricate case-based analysis which is often routine and repetitive. For example, in order to verify Theorem 1.4, AutoCase takes as input three symbolically-defined subsets R, G, B of the integers $[1, a^3(a + 1) - 1]$ (where $a \geq 7$ is a symbolic odd integer). AutoCase verifies that the union of R, G, and B is the set $[1, a^3(a + 1) - 1]$ and that R, G, B are mutually disjoint—thereby confirming that R, G, B form a 3-colouring of the positive

[1] github.com/laminazaman/RadoNumbers/tree/main/AutoCase

integers less than $a^3(a+1)$. Finally, AutoCase confirms that there are no monochromatic solutions of the equation $ax + ay = (a+1)z$, i.e., the set $\bigcup_{ax+ay=(a+1)z}\{(x, y, z)\}$ is disjoint with R^3, G^3, and B^3. We stress that the sets defining the colouring are defined *symbolically*, e.g., $\bigcup_{i,j,k}\{f(i, j, k)\}$ where f is a function (possibly depending on a), the upper and lower bounds on the indices i, j, k may depend on a, and the elements of the set may be filtered by divisibility predicates such as specifying that all elements are divisible by a.

Although it would be easy to perform the necessary set operations in a typical computer algebra system like Maple, Mathematica, and SageMath if a was a known fixed integer, in our application a is a symbolic variable, and we are aware of no computer algebra system supporting the operations on symbolic sets we require in this paper. Maple and Mathematica seem to lack the ability to even express the set $[1, a]$ when a is symbolic. SageMath does have support for a "ConditionSet" allowing sets like $[1, a]$ to be formulated, but the operations supported by ConditionSets was too limited for our purposes. In AutoCase, operations on symbolic sets are performed by employing a combination of the Python symbolic computation library `SymPy` [6] and the SMT (SAT modulo theories) constraint solver `Z3` [7].

Our work therefore fits into the "SC-Square" paradigm of combining satisfiability checking with symbolic computation [8, 9]. Over the past decade, the SC-Square community has combined the tools of satisfiability checking (e.g., SAT and SMT solvers) with the tools of symbolic computation (e.g., computer algebra systems and libraries) in order to make progress on problems benefiting from both the search and learning of SAT/SMT solvers and the mathematical sophistication of computer algebra. As just a single example, computer algebra libraries are able to detect if two mathematical objects are isomorphic. Augmenting a SAT solver with isomorphism detection can dramatically improve its efficiency on mathematical problems like proving the nonexistence of an order ten projective plane [10] or enumerating various kinds of graphs up to isomorphism, something that can be done by SAT modulo symmetries [11] or SAT+CAS [12] solvers.

2. Computational Results

One of the key challenges in Ramsey theory on the integers is the scarcity of data, as generating individual data points is an exceptionally difficult task. However, modern computational tools, including SAT solvers, have alleviated some of these computational challenges. During the last two decades, SAT solvers have been employed to compute Ramsey-type numbers at different scales and capacities. Some examples are the computation of the values and bounds of van der Waerden numbers (Kouril and Paul [13], Herwig et al. [14], Ahmed [15, 16, 17, 18], Ahmed et al. [19]) and Schur numbers (Ahmed and Schaal [20], Ahmed et al. [21]). A notable success of SAT solvers in this area of mathematics is the computation of the fifth Schur number by Heule [22].

Recently, Chang, De Loera, and Wesley [4] applied SAT solvers to compute Rado numbers. Building on this progress, we used SAT solvers to determine new exact values of Rado numbers and to uncover patterns in the colourings that avoid monochromatic solutions to the linear equation under investigation. These patterns provide general lower bounds on Rado numbers and we employed methods from symbolic computation and satisfiability checking to automate the verification of these lower bounds.

2.1. The satisfiability (SAT) problem

A *literal* is a Boolean variable (say x) or its negation (denoted $\bar{x}$). A *clause* is a logical disjunction of literals. A *formula* is in *Conjunctive Normal Form (CNF)* if it is a logical conjunction of clauses.

A *truth assignment* is a mapping of each variable in its domain to true or false. A truth assignment *satisfies a clause* if it maps at least one of its literals to true and the assignment *satisfies a formula* if it satisfies each of its clauses. A formula is called *satisfiable* if it is satisfied by at least one truth assignment and otherwise it is called *unsatisfiable*. The problem of recognizing satisfiable formulas is known as *the satisfiability problem*, or SAT for short.

2.2. Encoding Rado numbers as SAT problems

Given positive integers n and k and a linear equation $\mathcal{E}$, we now construct a formula in conjunctive normal form that is satisfiable if and only if there exists a k-colouring of $[1,n]$ avoiding monochromatic solutions of $\mathcal{E}$. Therefore, if our formula is satisfiable then $R_k(\mathcal{E}) > n$ and if our formula is unsatisfiable then $R_k(\mathcal{E}) \leq n$.

Variables: Variables are denoted by $v_{i,j}$ for $0 \leq i \leq k-1$ and $1 \leq j \leq n$, such that $v_{i,j}$ is true if and only if colour i is assigned to integer j. There are nk variables in the formula.

At least one colour is assigned to every integer: For each integer $j \in [1,n]$, the clause

$$(v_{0,j} \vee v_{1,j} \vee \cdots \vee v_{k-1,j})$$

ensures at least one colour $i \in [0,k-1]$ is assigned to j.

At most one colour is assigned to every integer: For each integer $j \in [1,n]$, the clauses

$$\bigwedge_{0\leq i_1<i_2\leq k-1} (\bar{v}_{i_1,j} \vee \bar{v}_{i_2,j})$$

ensure at most one colour $i \in [0,k-1]$ is assigned to j.

There is no monochromatic solution to $\mathcal{E}$: For each colour $i \in [0,k-1]$ and for each solution $(x_1,x_2,\ldots,x_m)$ to $\mathcal{E}$, the clause

$$(\bar{v}_{i,x_1} \vee \bar{v}_{i,x_2} \vee \cdots \vee \bar{v}_{i,x_m})$$

ensures the solution $(x_1,x_2,\ldots,x_m)$ is not monochromatic in colour i. Let $\mathcal{S}_{\mathcal{E},n}$ denote the set of all solutions $(x_1,x_2,\ldots,x_m)$ in $[1,n]$ that satisfy the equation $\mathcal{E}$. There will be $k \cdot |\mathcal{S}_{\mathcal{E},n}|$ such clauses. We may compute $|\mathcal{S}_{\mathcal{E},n}|$ by summing over all possible values of $(x_1,x_2,\ldots,x_{m-1})$ in the range $[1,n]$ via

$$|\mathcal{S}_{\mathcal{E},n}| = \sum_{x_1=1}^{n}\sum_{x_2=1}^{n}\cdots\sum_{x_{m-1}=1}^{n}\left[\frac{\sum_{i=1}^{m-1} a_i x_i}{a_m} \in [1,n]\right],$$

where summand uses Iverson bracket notation to ensure that the remaining variable x_m (determined by the equation) falls within $[1,n]$. This gives $|\mathcal{S}_{\mathcal{E},n}| \leq n^{m-1}$, since $[\cdot] \leq 1$.

Breaking permutation symmetry of colours: So far, no distinction has been made amongst the k colours. This artificially increases the size of the search space by a factor of $k!$, the number of permutations on k colours. To avoid having the SAT solver explore all 3! colour permutations on $\{0,1,2\}$ during solving, we ensure that colours are introduced in ascending order, i.e., colour 0 appears before colour 1 and colour 1 appears before colour 2. To ensure that colour 0 appears in the first position, we add the unit clause $v_{0,1}$. To ensure that colour 1 appears before colour 2, we add the clause $(\bigvee_{i=1}^{j-1} \bar{v}_{0,i}) \vee \bar{v}_{2,j}$ for $j = 2,\ldots,R_1(\mathcal{E})$. These say that colour 2 cannot immediately follow an initial colouring consisting only of colour 0s, and thus the first colour other than 0 must be 1. One can work out the exact value of $R_1(\mathcal{E})$ by hand for the cases we consider in this paper, as in the following lemma whose proof we provide in appendix A.

Lemma 2.1. *If a and b are positive coprime integers then $R_1(\mathcal{E}(3,0;a,b,b)) = \max(a+1,b)$, $R_1(\mathcal{E}(3,0;a,a,1)) = 2a$, and $R_1(\mathcal{E}(3,0;a,a,b)) = \max(a,\lceil b/2\rceil)$ when $b>1$.*

2.3. Efficient generation of integer solutions

To efficiently generate all integer solutions (x, y, z) within the interval $[1, n]$ for $ax + by = cz$, we iterate i from 1 to n and construct three linear Diophantine equations

$$\begin{aligned} ax &= cz - by \quad \text{where } x = i, \\ by &= cz - ax \quad \text{where } y = i, \\ cz &= ax + by \quad \text{where } z = i. \end{aligned}$$

The goal is to independently find solutions for each equation.

- Step 1 (Solving a linear Diophantine equation). To begin, we solve one of the above equations (say $ax + by = ci$) for unknowns (x, y) using the Extended Euclidean Algorithm, providing a particular integer solution (x_0, y_0). The general solution is

$$x = x_0 + \frac{bk}{\gcd(a,b)} \quad \text{and} \quad y = y_0 - \frac{ak}{\gcd(a,b)},$$

  where k is a parameter that runs over all integers to generate all solutions.
- Step 2 (Restricting to the interval $[1, n]$). Since we are only interested in solutions within $[1, n]$, we impose the constraints

$$1 \le x_0 + \frac{bk}{\gcd(a,b)} \le n \ \text{ and } \ 1 \le y_0 - \frac{ak}{\gcd(a,b)} \le n.$$

  These inequalities provide upper and lower bounds on k, ensuring that both x and y remain in the valid range.
- Step 3 (Iterating over valid values of k). Once the feasible range for k is determined, we iterate over all valid values of k and solve for x and y at each step, adding (x, y, i) to our list of solutions.
- Finally, we repeat these steps for the remaining two equations $ai = cz - by$ and $bi = cz - ax$.

2.4. SAT solvers and computation resources

For this work, we used the SAT solvers CaDiCaL [23] and Kissat [24]. Initially, we used CaDiCaL integrated with PySAT [25] due to its incremental solving capabilities which enable the solver to reuse learned information across instances. Our approach starts with a counter $n = 1$. For each n, we add the corresponding clauses to the solver. If the instance is satisfiable, we increment n and continue incrementally. This process repeats until we encounter the first unsatisfiable instance, at which point the corresponding n is identified as the Rado number.

Once a Rado number was determined (or suspected to be known based on the patterns we observed), we transitioned to using a non-incremental solver and generated two instances: one for $n - 1$, the last satisfiable instance, and one for n, the first unsatisfiable instance. We generated clauses in DIMACS format, wrote them to two CNF files, and used Kissat to solve both instances. To be consistent in our timings, all instances were ultimately solved non-incrementally using Kissat. Our largest instance, the unsatisfiable instance showing $R_3(\mathcal{E}(3, 0; 15, 15, 8)) = 97875$, contained 293,625 variables and 255,884,401 clauses and took 58.8 hours to solve.

Altogether, generating the SAT instances took around 11 hours, the total CPU time used to solve the unsatisfiable instances (i.e., the instances that prove an upper bound) was around 251 hours, and the total CPU time used on satisfiable instances (i.e., the instances that prove a lower bound) was around 53 hours. Our computations were performed on the Compute Canada cluster `Cedar`, utilizing Intel E5-2683 v4 Broadwell processors running at 2.1 GHz.

2.5. Some new exact values of $R_3(\mathcal{E}(3, 0; a, b, b))$

We have computed $R_3(\mathcal{E}(3, 0; a, b, b))$ for $1 \le b \le 15$ and $1 \le a \le 30$ and given the results in Table 1. By Rado's theorem 1.1, all elements in Table 1 are finite. Altogether, the total CPU time on unsatisfiable

$a \backslash b$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	14	14	27	64	125	216	343	512	729	1000	1331	1728	2197	2744	3375
2	43	14	31	14	125	27	343	64	729	125	1331	216	2197	343	3375
3	94	61	14	73	125	14	343	512	27	1000	1331	64	2197	2744	125
4	173	43	109	14	141	31	343	14	729	125	1331	27	2197	343	3375
5	286	181	186	180	14	241	343	512	729	14	1331	1728	2197	2744	27
6	439	94	43	61	300	14	379	73	31	125	1331	14	2197	343	125
7	638	428	442	456	470	462	14	561	729	1000	1331	1728	2197	14	3375
8	889	173	633	43	665	109	644	14	793	141	1331	31	2197	343	3375
9	1198	856	94	892	910	61	896	896	14	1081	1331	73	2197	2744	125
10	1571	286	1171	181	43	186	1190	180	1206	14	1431	241	2197	343	31
11	2014	1508	1530	1552	1574	1596	1618	1584	1575	1580	14	1849	2197	2744	3375
12	2533	439	173	94	2005	43	2053	61	109	300	2024	14	2341	379	141
13	3134	2432	2458	2484	2510	2536	2562	2588	2574	2530	2541	2544	14	2913	3375
14	3823	638	3039	428	3095	442	43	456	3207	470	3113	462	3146	14	3571
15	4606	3676	286	3736	94	181	3826	3856	186	61	3795	180	3835	3836	14
16	5489	889	**4465**	173	**4529**	633	**4593**	43	**4657**	665	**4576**	109	**4602**	644	**4620**
17	6478	**5288**	**5322**	**5356**	**5390**	**5424**	**5458**	**5492**	**5526**	**5560**	**5594**	**5424**	**5447**	**5474**	**5475**
18	7579	1198	439	856	**6355**	94	**6427**	892	43	910	**6571**	61	**6409**	896	300
19	8798	**7316**	**7354**	**7392**	**7430**	**7468**	**7506**	**7544**	**7582**	**7620**	**7658**	**7696**	**7488**	**7518**	**7530**
20	10141	1571	**8541**	286	173	1171	**8701**	181	**8781**	43	**8861**	186	**8941**	1190	109
21	11614	**9808**	638	**9892**	**9934**	428	94	**10060**	442	**10144**	**10186**	456	**10270**	61	470
22	13223	2014	**11287**	1508	**11375**	1530	**11463**	1552	**11551**	1574	43	1596	**11727**	1618	**11535**
23	14974	**12812**	**12858**	**12904**	**12950**	**12996**	**13042**	**13088**	**13134**	**13180**	**13226**	**13272**	**13318**	**13364**	**13110**
24	16873	2533	889	439	**14665**	173	**14761**	94	633	2005	**14953**	43	**15049**	2053	665
25	18926	**16376**	**16426**	**16476**	286	**16576**	**16626**	**16676**	**16726**	181	**16826**	**16876**	**16926**	**16976**	186
26	21139	3134	**18435**	2432	**18539**	2458	**18643**	2484	**18747**	2510	**18851**	2536	43	2562	**19059**
27	23518	**20548**	1198	**20656**	**20710**	856	**20818**	**20872**	94	**20980**	**21034**	892	**21142**	**21196**	910
28	26069	3823	**22933**	638	**23045**	3039	173	428	**23269**	3095	**23381**	442	**23493**	43	**23605**
29	28798	**25376**	**25434**	**25492**	**25550**	**25608**	**25666**	**25724**	**25782**	**25840**	**25898**	**25956**	**26014**	**26072**	**26130**
30	31711	4606	1571	3676	439	286	**28351**	3736	1171	94	**28591**	181	**28711**	3826	43

Table 1
$R_3(\mathcal{E}(3,0;a,b,b))$ for $1 \le b \le 15$ and $1 \le a \le 30$. The previously unknown values are presented in boldface, and the underlined entries correspond to equations whose coefficients are not coprime.

instances was 21,790 seconds, and the total CPU time on satisfiable instances was 13,516 seconds, amounting to 9.81 total hours. A maximum of 4.3 GiB of memory was used across all instances.

2.6. Some new exact values of $R_3(\mathcal{E}(3,0;a,a,b))$

By Theorem 1.2, $R_3(\mathcal{E}(3,0;a,a,1))$ is infinite for $a \ge 2$ and $R_3(\mathcal{E}(3,0;1,1,b))$ is infinite for $b \ge 4$. Also, by Chang et al. [4, Lemma 3.1], $R_3(\mathcal{E}(3,0;a,a,b))$ is infinite if $2a \le a^2/b$ or $2a \le \sqrt{ab}$. They provided the values of $R_3(\mathcal{E}(3,0;a,a,b))$ for $1 \le a, b \le 10$ and also for $3 \le a \le 6$ and $11 \le b \le 20$. We have extended them for $1 \le a \le 15$ and $1 \le b \le 25$ and reported these numbers in Table 2. The instances in this table proving upper bounds required a CPU time of 882,211 seconds to solve, while the instances proving lower bounds required 176,375 seconds to solve, amounting to 294.05 total hours. A maximum of 26.3 GiB of memory was used across all instances.

2.7. Some new patterns for $R_3(\mathcal{E}(3,0;a,b,b))$

The data on $R_3(\mathcal{E}(3,0;a,b,b))$ presented in Table 1 inspired us to make some general observations and a conjecture described below.

Observation 2.1. *For coprime integers a and b, the data in Table 1 reveals the following.*

$b\backslash a$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	14	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
2	1	$\underline{14}$	243	$\underline{\infty}$	∞	$\underline{\infty}$	∞	$\underline{\infty}$	∞	$\underline{\infty}$	∞	$\underline{\infty}$	∞	$\underline{\infty}$	∞
3	54	54	$\underline{14}$	384	2000	$\underline{\infty}$	∞	∞	$\underline{\infty}$	∞	∞	$\underline{\infty}$	∞	∞	$\underline{\infty}$
4	∞	$\underline{1}$	108	$\underline{14}$	875	$\underline{243}$	4459	$\underline{\infty}$	∞	$\underline{\infty}$	∞	$\underline{\infty}$	∞	$\underline{\infty}$	∞
5	∞	105	135	180	$\underline{14}$	864	3430	3072	12393	$\underline{\infty}$	∞	∞	∞	∞	$\underline{\infty}$
6	∞	$\underline{54}$	$\underline{1}$	$\underline{54}$	750	$\underline{14}$	3087	$\underline{384}$	$\underline{243}$	$\underline{2000}$	**27951**	$\underline{\infty}$	∞	$\underline{\infty}$	$\underline{\infty}$
7	∞	455	336	308	875	756	$\underline{14}$	1536	8748	7500	**23958**	**10368**	**54925**	$\underline{\infty}$	∞
8	∞	$\underline{\infty}$	432	$\underline{1}$	1000	$\underline{108}$	2744	$\underline{14}$	8019	$\underline{875}$	**21296**	$\underline{243}$	**48334**	$\underline{4459}$	**97875**
9	∞	∞	$\underline{54}$	585	1125	$\underline{54}$	3087	1224	$\underline{14}$	6000	**18634**	$\underline{384}$	**41743**	**30184**	$\underline{2000}$
10	∞	$\underline{\infty}$	1125	$\underline{105}$	$\underline{1}$	$\underline{135}$	3430	$\underline{180}$	7290	$\underline{14}$	**17303**	$\underline{864}$	**37349**	$\underline{3430}$	$\underline{243}$
11	∞	∞	2019	847	1958	1188	**3773**	**1672**	**8019**	**5500**	$\underline{14}$	**6048**	**35152**	**24696**	**77625**
12	∞	$\underline{\infty}$	$\underline{\infty}$	$\underline{54}$	2400	$\underline{1}$	**4116**	$\underline{54}$	$\underline{108}$	$\underline{750}$	**15972**	$\underline{14}$	**32955**	$\underline{3087}$	$\underline{875}$
13	∞	∞	∞	1710	3445	1963	**4459**	**1456**	**9477**	**6500**	**17303**	**5616**	$\underline{14}$	**21952**	**60750**
14	∞	$\underline{\infty}$	∞	$\underline{455}$	3675	$\underline{336}$	$\underline{1}$	$\underline{308}$	**10206**	$\underline{875}$	**18634**	$\underline{756}$	**30758**	$\underline{14}$	**57375**
15	∞	∞	$\underline{\infty}$	5408	$\underline{54}$	$\underline{105}$	**6615**	**2760**	$\underline{135}$	$\underline{54}$	**19965**	$\underline{180}$	**32955**	**20580**	$\underline{14}$
16	∞	$\underline{\infty}$	∞	$\underline{\infty}$	5725	$\underline{432}$	**7616**	$\underline{1}$	**11664**	$\underline{1000}$	**21296**	$\underline{108}$	**35152**	$\underline{2744}$	**54000**
17	∞	∞	∞	∞	8330	4743	**10064**	**3825**	**12393**	**8500**	**22627**	**7344**	**37349**	**23324**	**57375**
18	∞	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$	12069	$\underline{54}$	**10962**	$\underline{585}$	$\underline{1}$	$\underline{1125}$	**23958**	$\underline{54}$	**39546**	$\underline{3087}$	$\underline{750}$
19	∞	∞	∞	∞	16397	6726	**14782**	**4332**	**16853**	**9500**	**25289**	**8208**	**41743**	**26068**	**60750**
20	∞	$\underline{\infty}$	∞	$\underline{\infty}$	$\underline{\infty}$	$\underline{1125}$	**14700**	$\underline{105}$	**19080**	$\underline{1}$	**26620**	$\underline{135}$	**43940**	$\underline{3430}$	$\underline{108}$
21	∞	∞	$\underline{\infty}$	∞	∞	$\underline{455}$	$\underline{54}$	**6699**	$\underline{336}$	**12579**	**27951**	$\underline{308}$	**46137**	$\underline{54}$	$\underline{875}$
22	∞	$\underline{\infty}$	∞	$\underline{\infty}$	∞	$\underline{\mathbf{2019}}$	**20580**	$\underline{\mathbf{847}}$	**25047**	$\underline{\mathbf{1958}}$	$\underline{1}$	$\underline{\mathbf{1188}}$	**48334**	$\underline{\mathbf{3773}}$	**74250**
23	∞	∞	∞	∞	∞	**19056**	**27853**	**8556**	**32453**	**17250**	**35926**	**9936**	**50531**	**31556**	**77625**
24	∞	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$	∞	$\underline{\infty}$	**28956**	$\underline{54}$	$\underline{432}$	$\underline{\mathbf{2400}}$	**39600**	$\underline{1}$	**52728**	$\underline{\mathbf{4116}}$	$\underline{1000}$
25	∞	∞	∞	∞	$\underline{\infty}$	∞	**40163**	**10200**	**42975**	$\underline{105}$	**47850**	**12850**	**54925**	**34300**	$\underline{135}$

Table 2
$R_3(\mathcal{E}(3,0;a,a,b))$ for $1 \le a \le 15$ and $1 \le b \le 25$. The previously unknown values are presented in boldface, and the underlined entries correspond to equations whose coefficients are not coprime.

b	Range of a	$R_3(\mathcal{E}(3,0;a,b,b))$
2	$[7,30]$	a^3+a^2+5a+1
3	$[4,30]$	a^3+a^2+7a+1
4	$[7,30]$	a^3+a^2+9a+1
5	$[7,30]$	$a^3+a^2+11a+1$
6	$[11,30]$	$a^3+a^2+13a+1$
7	$[11,30]$	$a^3+a^2+15a+1$
8	$[13,30]$	$a^3+a^2+17a+1$

b	Range of a	$R_3(\mathcal{E}(3,0;a,b,b))$
9	$[14,30]$	$a^3+a^2+19a+1$
10	$[17,30]$	$a^3+a^2+21a+1$
11	$[17,30]$	$a^3+a^2+23a+1$
12	$[19,30]$	$a^3+a^2+25a+1$
13	$[20,30]$	$a^3+a^2+27a+1$
14	$[23,30]$	$a^3+a^2+29a+1$
15	$[26,30]$	$a^3+a^2+31a+1$

The formulas presented above were derived through curve fitting. Our analysis began by identifying patterns within each column of data. Some numbers exhibited a constant difference, suggesting a linear relationship, while others showed a constant third difference, indicating a cubic relationship. We constructed systems of linear and cubic equations based on these observations. We then applied Gaussian–Jordan elimination to solve these systems, obtaining the best-fitting formulas describing the sequence and ultimately arriving at the following conjecture.

Conjecture 2.1. *For coprime positive integers a and b with $a^2+a+b > b^2+ba$ and $a > b \ge 3$, $R_3(\mathcal{E}(3,0;a,b,b)) = a^3+a^2+(2b+1)a+1$.*

Before proceeding, we note the equation $\mathcal{E}(3,0;a,-a,b)$ can be transformed into $\mathcal{E}(3,0;b,a,a)$ by permuting variables. This equivalence along with a theorem of Chang, De Loera, and Wesley [4] provides a closed-form expression for $R_3(\mathcal{E}(3,0;a,1,1))$.

Proposition 2.1. $R_3(\mathcal{E}(3,0;a,1,1)) = a^3+5a^2+7a+1$ *for* $a \ge 1$.

Proof. By permuting variables we have $R_3(\mathcal{E}(3,0;a,1,1)) = R_3(\mathcal{E}(3,0;1,-1,a))$. By [4, Thm 1.2], for $m \ge 3$, $R_3(\mathcal{E}(3,0;1,-1,m-2)) = m^3-m^2-m-1$. Replacing a with $m-2$, we get for $a \ge 1$, $R_3(\mathcal{E}(3,0;1,-1,a)) = a^3+5a^2+7a+1$. Hence, $R_3(\mathcal{E}(3,0;a,1,1)) = a^3+5a^2+7a+1$. ∎

2.8. Some new patterns for $R_3(\mathcal{E}(3,0;a,a,b))$

The data on $R_3(\mathcal{E}(3,0;a,a,b))$ presented in Table 2 has inspired the following conjecture and some general observations described below.

Conjecture 2.2. *For coprime positive integers a and b, if a is odd, then for $3 \le a < b \le 2a-1$, $R_3(\mathcal{E}(3,0;a,a,b)) = a^3 b$.*

Observation 2.2. *For odd positive integers $7 \le a \le 15$, we have*
- $R_3(\mathcal{E}(3,0;a,a,a-1)) = R_3(\mathcal{E}(3,0;a,a,a+2)) = a^3(a+2)$*;*
- *if* $\gcd(a,a+3)=1$*, then* $R_3(\mathcal{E}(3,0;a,a,a-2)) = R_3(\mathcal{E}(3,0;a,a,a+3)) = a^3(a+3)$*;*
- $R_3(\mathcal{E}(3,0;a,a,\frac{a+1}{2})) = R_3(\mathcal{E}(3,0;a,a,2a-1)) = a^3(2a-1)$*;*
- *if* $\gcd(a,\frac{a+3}{2})=1$*, then* $R_3(\mathcal{E}(3,0;a,a,\frac{a+3}{2})) = R_3(\mathcal{E}(3,0;a,a,2a-4)) = a^3(2a-4)$*.*

Observation 2.3. *For coprime positive integers a and b,*

$$R_3(\mathcal{E}(3,0;a,a,b)) = \begin{cases} a^3b/2 & \text{if } a \in \{6,10,14\} \text{ and } a < b \le 2a-1, \\ a^3b/4 & \text{if } a \in \{12\} \text{ and } a < b \le 2a-1. \end{cases}$$

A general characterization of the cases with even a is not possible at the moment due to lack of further data.

3. General Results

In Section 2, we extended the known numerical values of Rado numbers corresponding to the equations $\mathcal{E}(3,0;a,a,b)$ and $\mathcal{E}(3,0;a,b,b)$ and observed several new bounds for these numbers. In this section, we discuss and prove some of those bounds using AutoCase. An overview of how AutoCase works is first provided in Section 3.1, and then as a demonstrative example we use AutoCase to prove Proposition 2.1 in Section 3.2. Finally, we use AutoCase to prove Theorems 1.3 and 1.4 in Sections 3.3 and 3.4.

3.1. Automating case-based analysis with `SymPy` and `Z3`

Shallit [26] proposed that traditional case-by-case analyses are better performed by automated search algorithms to enhance both efficiency and correctness. In that direction, we employ an automated verification approach (using our tool AutoCase) to verify the case-based proofs of Theorems 1.3 and 1.4. Our proof strategy relies on case-based analysis of solving linear equations whose variables lie in symbolic sets filtered through arithmetic and logical constraints. This section describes how we automate this process by symbolically encoding the problem structure and constraint system using `SymPy` and performing satisfiability checking using the SMT solver `Z3`.

The input provided to AutoCase is the linear equation to be considered along with a colouring of the integers $[1,N]$ specified by a collection of symbolic sets as described in Section 3.1.1. AutoCase verifies that the colouring is valid by ensuring the size of the symbolic sets sum to exactly N (see Section 3.1.2) and that the sets are pairwise disjoint (see Section 3.1.3). This verifies the provided symbolic sets provide a partition of $[1,N]$. Finally, AutoCase verifies that there are no solutions of the linear equation under consideration where all variables are from symbolic sets of the same colour (see Section 3.1.4). Thus, the symbolic sets define a colouring of the integers $[1,N]$ that have no monochromatic solutions to the linear equation, thereby proving a lower bound of $N+1$ on the Rado number for that linear equation.

3.1.1. Input specification

The input to our automated system consists of:
- A symbolic linear equation $\mathcal{E}$, such as $ax+ay=(a+1)z$, where coefficients and assumptions on the coefficients (e.g., $a \ge 7$, a odd) are symbolic.
- A symbolic N corresponding to the lower bound on $R_k(\mathcal{E})$, e.g., $N = a^3(a+1)-1$.

- A partition of the domain $[1, N]$ into named symbolic sets, each specified by
  - symbolic bounds (e.g., $[a, a^4 + a^3 - a]$),
  - a format expression generating integers in the set (e.g., $a(a+1)i + aj + k$), and
  - optional divisibility filters (e.g., integers in the set must be divisible by a^2 but not a^3).
- A k-colouring of $[1, N]$ defined in terms of the above symbolic sets.

These inputs are defined using `SymPy` to facilitate symbolically:
- Building and manipulating (e.g., simplifying, expanding, combining, substituting symbolically) interval bounds, format expressions, and divisibility properties.
- Calculating size, disjointness, and covering conditions. The union $\bigcup_{i=1}^{k} S_i$ is considered a partition of $[1, N]$ if and only if $\bigcup_{i=1}^{k} S_i$ has size N, the sets S_i are pairwise disjoint, and $\bigcup_{i=1}^{k} S_i \subseteq [1, N]$.
- Simplifying and structurally transforming constraints before they are fed to `Z3`.

3.1.2. Symbolic size computation using `SymPy`

`SymPy` plays a central role in enabling symbolic reasoning, algebraic manipulation, and structured enumeration in the symbolic size analysis pipeline. The key uses of `SymPy` can be itemized as follows:
- **Size with divisibility filters**: Let the underlying interval be defined as $[L, H]$. Let D and ND denote the sets of required and excluded divisors, respectively, and consider

  $$S = \{\, x \in [L, H] : \forall d \in D, d \mid x \text{ and } \forall n \in ND, n \nmid x \,\}.$$

  For $T \subseteq ND$, define

  $$S_T = \{\, x \in [L, H] : \operatorname{lcm}(D \cup T) \mid x \,\}.$$

  The number of elements $x \in [L, H]$ divisible by all divisors in D and none in ND is

  $$|S| = \sum_{T \subseteq ND} (-1)^{|T|} |S_T| = \sum_{T \subseteq ND} (-1)^{|T|} \left(\left\lfloor \frac{H}{\operatorname{lcm}(D \cup T)} \right\rfloor - \left\lfloor \frac{L-1}{\operatorname{lcm}(D \cup T)} \right\rfloor \right).$$

  This formula uses inclusion-exclusion to subtract overlapping exclusions in order to count elements satisfying the divisibility constraints. `SymPy` contributes with `floor`, `ceiling`, and `lcm` to enforce and combine arithmetic conditions. For example, consider the interval $[1, b^2 a]$ with $D = \{b\}$ and $ND = \{b^2\}$. Then the size of the resulting set is

  $$\left\lfloor \frac{b^2 a}{\operatorname{lcm}(b)} \right\rfloor - \left\lfloor \frac{b^2 a}{\operatorname{lcm}(b, b^2)} \right\rfloor = \left\lfloor \frac{b^2 a}{b} \right\rfloor - \left\lfloor \frac{b^2 a}{b^2} \right\rfloor = ba - a = a(b-1).$$

- **Format-based symbolic summation**: Consider a symbolic set defined by a format expression $x = f(i_1, i_2, \dots, i_k)$ where $i_j \in [l_j, h_j]$. When f is injective, the total size of the set is given by

  $$|\{x = f(i_1, \dots, i_k) \mid l_j \le i_j \le h_j\}| = \sum_{i_1 = l_1}^{h_1} \sum_{i_2 = l_2}^{h_2} \cdots \sum_{i_k = l_k}^{h_k} 1.$$

  For example, the format expression $x = ai + j$ where $0 \le i < m$ and $1 \le j \le a$ defines a set of size ma. If some variables are symbolic (e.g., a), the size remains symbolic and can be simplified using algebraic rules. For example, $\sum_{i=0}^{a} \sum_{j=1}^{i} 1 = a(a+1)/2$.
- **Symbolic floor simplification**: Let a be a symbolic variable representing a positive integer, and consider the rational expression $f(a) = p(a)/q(a)$, where p and q are polynomials in a. We aim to simplify $\lfloor p(a)/q(a) \rfloor$ which can often be simplified using the degrees of the numerator and denominator polynomials.
  Consider $\deg(p) < \deg(q)$. When the degree of the numerator $p(a)$ is strictly less than that of the denominator $q(a)$, the value of the rational expression $p(a)/q(a)$ has magnitude less than 1 for all

sufficiently large a, and is dominated by the leading terms of p and q. Its sign for large enough a is therefore determined by the sign of $\mathrm{LC}(p)/\mathrm{LC}(q)$ where $\mathrm{LC}(\cdot)$ denotes the leading coefficient, so

$$\left\lfloor \frac{p(a)}{q(a)} \right\rfloor = \begin{cases} 0 & \text{if } \mathrm{LC}(p)/\mathrm{LC}(q) > 0, \\ -1 & \text{if } \mathrm{LC}(p)/\mathrm{LC}(q) < 0, \end{cases}$$

assuming that the bound on a is large enough for this simplification to occur. For example, $\lfloor -1/a \rfloor = -1$ if $a \geq 1$, $\lfloor 1/a \rfloor = 0$, $\lfloor (a+1)/a \rfloor = 1$, and $\lfloor (a^2+1)/a^3 \rfloor = 0$ if $a \geq 2$, and $\lfloor -(a+5)/a^2 \rfloor = -1$ if $a \geq 3$.

- **Unified symbolic simplification**: Applies `simplify` and `expand` to normalize symbolic size expressions such as converting $2a(a+1) + a(a^2+3a+1) + 4a$ to $a^3 + 5a^2 + 7a$.

3.1.3. Symbolic disjointness checking using `SymPy` and `Z3`

AutoCase constructs a symbolic system of constraints to determine whether two symbolic sets are disjoint. The key idea is to introduce a symbolic integer variable z that hypothetically belongs to both sets. Then, disjointness reduces to checking whether the conjunction of the constraints defining both sets is unsatisfiable. `Z3` is used to test satisfiability.

- **Domain constraints (bounds)**: For each interval, we require $z \in [\texttt{lower}, \texttt{upper}]$, encoded via `Ge(z, lower)` and `Le(z, upper)`. If sets A and B are defined over $[1, a^2+2a]$ and $[a^2+3a+1, a^3+5a^2+7a]$, respectively, then

$$z \geq 1 \wedge z \leq a^2 + 2a \quad \text{and} \quad z \geq a^2 + 3a + 1 \wedge z \leq a^3 + 5a^2 + 7a.$$

  These bounds are incompatible, so A and B are disjoint.
- **Divisibility and non-divisibility constraints**: Divisibility constraints are encoded as $d \mid z$ using $\texttt{Mod}(z, d) == 0$, and non-divisibility as $d \nmid z$ using $\texttt{Mod}(z, d) \neq 0$. Suppose set A contains all $z \in [1, ab]$ such that $b \mid z$, while set B contains those where $b \nmid z$. The constraint system $z \equiv 0 \mod b \wedge z \not\equiv 0 \mod b$ is unsatisfiable, so the sets are disjoint.
- **Format expression constraints**: To avoid symbol collisions in format expressions, all format variables are renamed using fresh symbols (e.g., `_fmt_i`). If a set is defined by a format expression $z = f(i_1, \ldots, i_k)$, then we add `sp.Eq(z, format_expr)` along with symbolic bounds on the variables i_j (e.g., $0 \leq i < a^2$).
  For instance, consider the set A defined by

$$z = a(a+1)i + aj + k \quad \text{where} \quad 0 \leq i < a^2, \quad 0 \leq j \leq a, \quad \text{and} \quad 2\lfloor j/2 \rfloor + 1 \leq k \leq a - 1,$$

  with constraint $z \not\equiv 0 \pmod a$. Also, let the set B be defined by

$$z = t(a^3 + a^2), \quad 1 \leq t \leq \lfloor (a-1)/2 \rfloor.$$

  Now, assume $z \in A \cap B$. Then $z \in B$ implies $z \equiv 0 \pmod a$, and this contradicts the constraint from A. Therefore, $A \cap B = \emptyset$.

3.1.4. Monochromatic solution analysis using `Z3`

For a linear equation $\sum_{i=1}^{m-1} a_i x_i = a_m x_m$ and $t \geq 1$ sets (each given a colour class), the tool now generates all possible ways to assign the $m-1$ left-hand-side (LHS) variables across t sets using Cartesian products. AutoCase iterates over every way to assign the LHS variables to the same colour class and for every colour class it generates a list of cases for verification.

The prover systematically explores all combinations of sets assigned to the variables in the equation, where each variable is taken from the same colour class. For each such case, it verifies that no solution exists satisfying the equation under the imposed divisibility, format, and value constraints. If all such cases succeed this confirms the absence of monochromatic solutions.

- For each assignment of sets to the variables in the equation, the tool aggregates all associated symbolic constraints: interval bounds (e.g., $x \in [1, a^3]$), format constraints (e.g., $x = ia^2 + ja$), and arithmetic filters (e.g., $y \nmid a$, $z \mid a^2$).
- It then encodes the target equation (e.g., $ax + ay = (a+1)z$) and all constraints into an SMT expression, along with assumptions (e.g., $a \geq 7$, $\texttt{prime}(a)$, $\gcd(a, b) = 1$). The translation process supports a broad range of symbolic constructs in `SymPy`, including integer inequalities, polynomial equalities, floor and ceiling operations, divisibility conditions, and logical connectives.
- The solver checks that the symbolic system is unsatisfiable. The contradiction analysis is exhaustive—it verifies that under all parameter values satisfying the assumptions, the given assignment cannot yield a valid solution.

If every such assignment leads to a contradiction, the tool concludes that the equation has no monochromatic solution in the constructed colouring, and thus the lower bound for the Rado number is symbolically proven.

Example 3.1. *We illustrate a symbolic case analyzed by AutoCase, where each variable in the equation $ax_1 + ax_2 = (a+1)x_3$ is assigned to distinct symbolic intervals with bounds, divisibility filters, and format expressions. The analyzer constructs the following constraint system:*

- ***Global Assumptions**: $a \in \mathbb{Z}$ and $a \geq 1$*
- ***Equation Constraint**: $ax_1 + ax_2 = (a+1)x_3$*
- ***Bounds**: $x_1, x_2, x_3 \in [1, a^2] \quad \Rightarrow \quad x_i \geq 1, x_i \leq a^2$ for $i \in \{1, 2, 3\}$*
- ***Divisibility Filter** on x_1: $a \mid x_1 \quad \Rightarrow \quad x_1 = a \cdot k_1, \quad k_1 > 0$*
- ***Non-divisibility Filter** on x_2: $a^2 \nmid x_2 \quad \Rightarrow \quad x_2 \neq a^2 \cdot k_2$ for all $k_2 > 0$*
- ***Format Expression** for x_3:*

$$x_3 = a(a+1)i + aj + k \quad \textit{with} \quad 0 \leq i < a^2, 0 \leq j \leq a, \textit{ and } 2\lfloor j/2 \rfloor + 1 \leq k \leq a - 1$$

- ***Total Constraint Set** (passed to `Z3`):*

$$\begin{aligned}
&ax_1 + ax_2 = (a+1)x_3 \\
&x_1 = a \cdot k_1, \quad k_1 > 0 \\
&\forall k_2 > 0, x_2 \neq a^2 \cdot k_2 \\
&x_3 = a(a+1)i + aj + k \\
&0 \leq i < a^2, \quad 0 \leq j \leq a, \quad 2\lfloor j/2 \rfloor + 1 \leq k \leq a - 1 \\
&x_1, x_2, x_3 \in [1, a^2]
\end{aligned}$$

The set of constraints in `Z3` are given in Appendix D. If `Z3` determines that this system is unsatisfiable for all admissible values of a, then this configuration leads to a contradiction—contributing to the overall proof that the equation has no monochromatic solution in the constructed colouring.

3.2. Proving with AutoCase: An example

A general technique to prove a lower bound is to construct a colouring avoiding forbidden patterns. Specifically, to show $R_k(\mathcal{E}) \geq N + 1$, constructing a k-colouring of the positive integers $[1, N]$ avoiding any monochromatic solution to equation $\mathcal{E}$ would be sufficient. As an example, Proposition 2.1 states $R_3(\mathcal{E}(3, 0; a, 1, 1)) = a^3 + 5a^2 + 7a + 1$. Although we already showed this proposition is a consequence of a theorem of Chang, De Loera, and Wesley [4], as an example we now show how a form of Proposition 2.1 (with the equality replaced by a lower bound) can be proven using AutoCase.

3.2.1. Construction of the partition

Based on empirical observations of solutions to small instances of the problem, using certificates (satisfying assignments) generated by the SAT solver CaDiCaL, we identify a partition of $[1, a^3 + 5a^2 + 7a]$ into seven intervals $P_0, \ldots, P_6$ along with a corresponding colouring function δ.

Consider the partition of $[1, a^3 + 5a^2 + 7a]$ using the intervals

$$\begin{aligned}
P_0 &= [1, a], \\
P_1 &= [a + 1, a^2 + 2a], \\
P_2 &= [a^2 + 2a + 1, a^2 + 3a], \\
P_3 &= [a^2 + 3a + 1, a^3 + 4a^2 + 4a], \\
P_4 &= [a^3 + 4a^2 + 4a + 1, a^3 + 4a^2 + 5a], \\
P_5 &= [a^3 + 4a^2 + 5a + 1, a^3 + 5a^2 + 6a], \\
P_6 &= [a^3 + 5a^2 + 6a + 1, a^3 + 5a^2 + 7a].
\end{aligned}$$

From these, we define the colouring function $\delta\colon [1, a^3 + 5a^2 + 7a] \to [0, 1, 2]$ as

$$\delta(i) = \begin{cases} 0 & \text{if } i \in P_0 \cup P_2 \cup P_4 \cup P_6, \\ 1 & \text{if } i \in P_1 \cup P_5, \\ 2 & \text{if } i \in P_3. \end{cases}$$

3.2.2. Input processing

The prover considers all possible pairs x, y drawn from the same colour class, and verifies that $z = ax+y$ cannot belong to any set in the same colour class. Based on the symbolic colouring provided by δ, the prover enumerates all ways of assigning x, y, and z to the same colour. Specifically, it generates $4^3 = 64$ cases for colour 0, $2^3 = 8$ cases for colour 1, and $1^3 = 1$ case for colour 2, yielding a total of 73 cases. These cases correspond to the combinations of source intervals for (x, y) and target intervals for z.

3.2.3. Monochromaticity analysis

Example 3.2. *We check the cases where $x \in P_0$ and $y \in P_0$. The computed $z = ax + y$ must not belong to any of P_0, P_2, P_4, P_6. The reasons why z cannot belong to any of P_0, P_2, P_4, P_6 are as follows:*

- *Lower bound of z: Since $x, y \in P_0 = [1, a]$, their smallest possible values are $x = 1$ and $y = 1$. This gives $z = ax + y \geq a(1) + 1 = a + 1$.*
- *Upper bound of z: The largest values in P_0 are $x = a$ and $y = a$. This gives $z = ax + y \leq a(a) + a = a^2 + a$.*
- *Now, we check whether z can belong to any forbidden partition:*
  - $P_0 = [1, a]$*: Since $z \geq a + 1$, $z \notin P_0$.*
  - $P_2 = [a^2 + 2a + 1, a^2 + 3a]$*: The maximum possible z is $a^2 + a$, which is less than* $\min(P_2) = a^2 + 2a + 1$*. Thus, $z \notin P_2$.*
  - $P_4 = [a^3 + 4a^2 + 4a + 1, a^3 + 4a^2 + 5a]$*: Clearly, $z \leq a^2 + a$ is much smaller than* $\min(P_4)$*, so $z \notin P_4$.*
  - $P_6 = [a^3 + 5a^2 + 6a + 1, a^3 + 5a^2 + 7a]$*: Again, z is far smaller than* $\min(P_6)$*, so $z \notin P_6$.*

Since z is forced outside the monochromatic partitions, we conclude that no monochromatic solution exists in colour 0 when $x \in P_0$ and $y \in P_0$.

The same type of analysis applies to all other colour cases, ensuring that the generated constraints exhaustively eliminate any monochromatic solution. As a result, AutoCase has shown that $R_3(\mathcal{E}(3, 0; a, 1, 1)) \geq a^3 + 5a^2 + 7a + 1$ for all $a \geq 1$.

3.3. Proof of Theorem 1.3 (lower bound for $R_3(\mathcal{E}(3, 0; a, b, b))$)

3.3.1. Construction of the partition

Since $\gcd(a, b) = 1$, the variable z to be an integer in the equation $ax + by = bz$, the variable x must be a multiple of b. Consider the partition of $[1, a^3 + a^2 + (2b + 1)a]$ using the intervals

$$
\begin{aligned}
P_0 &= [1, ba],\\
P_1 &= [ba + 1, b^2 a],\\
P_2 &= [b^2 a + 1, ba^2 + ba],\\
P_3 &= [ba^2 + ba + 1, a^3 + a^2 + (b + 1)a], \text{ and}\\
P_4 &= [a^3 + a^2 + (b + 1)a + 1, a^3 + a^2 + (2b + 1)a].
\end{aligned}
$$

Note that indeed $[1, a^3 + a^2 + (2b+1)a] = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4$ and $P_i \cap P_j = \emptyset$ for $i \neq j$. Define the following sets to filter the above intervals for colouring in Dark Gray (0), Red (1), and Blue (2):

$$
\begin{aligned}
R_1 &= \{v \in P_0 : v \not\equiv 0 \pmod{b}\}\\
R_2 &= \left\{v \in P_0 \cup P_1 : v \equiv 0 \pmod{b^2}\right\}\\
R_3 &= P_4\\
B_1 &= \left\{ v \in P_0 \cup P_1 : \begin{array}{l} v \equiv 0 \pmod{b}, \\ v \not\equiv 0 \pmod{b^2} \end{array} \right\}\\
B_2 &= \{v \in P_2 : v \equiv 0 \pmod{b}\}\\
D_1 &= \{v \in P_1 \cup P_2 : v \not\equiv 0 \pmod{b}\}\\
D_2 &= P_3
\end{aligned}
$$

Now, consider the colouring $\delta\colon [1, a^3 + a^2 + (2b + 1)a] \to [0, 1, 2]$ defined by

$$
\delta(i) = \begin{cases} 0 & \text{if } i \in D_1 \cup D_2, \\ 1 & \text{if } i \in R_1 \cup R_2 \cup R_3, \\ 2 & \text{if } i \in B_1 \cup B_2. \end{cases}
$$

Example 3.3 provides a visual representation of this colouring.

Example 3.3. *Below is the colouring of* $[1, 108]$ *used to prove that* $R_3(\mathcal{E}(3, 0; 4, 3, 3)) \geq 109$.

r1	*r2*	*b3*
r4	*r5*	*b6*
r7	*r8*	*r9*
r10	*r11*	*b12*
P_0		

13	*14*	*b15*
16	*17*	*r18*
19	*20*	*b21*
22	*23*	*b24*
25	*26*	*r27*
28	*29*	*b30*
31	*32*	*b33*
34	*35*	*r36*
P_1		

37	*38*	*b39*
40	*41*	*b42*
43	*44*	*b45*
46	*47*	*b48*
49	*50*	*b51*
52	*53*	*b54*
55	*56*	*b57*
58	*59*	*b60*
P_2		

61	*62*	*63*
64	*65*	*66*
67	*68*	*69*
70	*71*	*72*
73	*74*	*75*
76	*77*	*78*
79	*80*	*81*
82	*83*	*84*
85	*86*	*87*
88	*89*	*90*
91	*92*	*93*
94	*95*	*96*
P_3		

r97	*r98*	*r99*
r100	*r101*	*r102*
r103	*r104*	*r105*
r106	*r107*	*r108*
P_4		

Proof of Theorem 1.3. A written proof of Theorem 1.3 is provided in Appendix B. The symbolic correctness of the partition and the verification of the cases can be found in the AutoCase repository. $\square$

3.4. Proof of Theorem 1.4 (lower bound for $R_3(\mathcal{E}(3,0;a,a,a+1))$)

Observation 3.1. *For odd positive integers $a \geq 7$, there exists a certificate (computed using a SAT solver) for $R_3(\mathcal{E}(3,0;a,a,a+1)) \geq a^3(a+1)$ of length $a^3(a+1)-1$ where certain repetitive patterns appear as blocks of coloured integers of size $a(a+1)$. Consider the following examples for $R_3(\mathcal{E}(3,0;7,7,8))$.*

r1	r2	r3	r4	r5	r6	7
r8	r9	r10	r11	r12	r13	14
b15	b16	r17	r18	r19	r20	21
b22	b23	r24	r25	r26	r27	28
b29	b30	b31	b32	r33	r34	35
b36	b37	b38	b39	r40	r41	42
b43	b44	b45	b46	b47	b48	r49
b50	b51	b52	b53	b54	b55	56

Colour block 1 for $R_3(\mathcal{E}(3,0;7,7,8))$

r57	r58	r59	r60	r61	r62	63
r64	r65	r66	r67	r68	r69	70
b71	b72	r73	r74	r75	r76	77
b78	b79	r80	r81	r82	r83	84
b85	b86	b87	b88	r89	r90	91
b92	b93	b94	b95	r96	r97	r98
b99	b100	b101	b102	b103	b104	105
b106	b107	b108	b109	b110	b111	112

Colour-block 2 for $R_3(\mathcal{E}(3,0;7,7,8))$

r337	r338	r339	r340	r341	r342	b343
r344	r345	r346	r347	r348	r349	350
b351	b352	r353	r354	r355	r356	357
b358	b359	r360	r361	r362	r363	364
b365	b366	b367	b368	r369	r370	371
b372	b373	b374	b375	r376	r377	378
b379	b380	b381	b382	b383	b384	385
b386	b387	b388	b389	b390	b391	r392

Colour-block 7 for $R_3(\mathcal{E}(3,0;7,7,8))$

r2689	r2690	r2691	r2692	r2693	r2694	b2695
r2696	r2697	r2698	r2699	r2700	r2701	2702
b2703	b2704	r2705	r2706	r2707	r2708	2709
b2710	b2711	r2712	r2713	r2714	r2715	2716
b2717	b2718	b2719	b2720	r2721	r2722	2723
b2724	b2725	b2726	b2727	r2728	r2729	2730
b2731	b2732	b2733	b2734	b2735	b2736	2737
b2738	b2739	b2740	b2741	b2742	b2743	

Colour-block 49 for $R_3(\mathcal{E}(3,0;7,7,8))$

The integers that are not divisible by a get coloured in a consistent way in each block, while the integers that are divisible by a need to be carefully coloured in such a way (a different way in each block) to avoid monochromatic solutions to the equation $ax+ay=(a+1)z$.

3.4.1. Construction of the partition

Before we prove the lower bound, we construct a partition of the set $[1,N]$ where $N=a^3(a+1)-1$ with $a \geq 7$ being an odd positive integer. Consider,

$$[1,N+1]=\left\{i\cdot a(a+1)+a\cdot j+k : 0\leq i\leq a^2-1, 0\leq j\leq a, 1\leq k\leq a\right\}.$$

Let $S_d(N)$ and $\overline{S_d}(N)$ denote $\{x\in[1,N] : d \mid x\}$ and $\{x\in[1,N] : d\nmid x\}$, respectively. We have the partition $[1,N]=S_a(N)\cup\overline{S_a}(N)$ with

$$S_a(N)=\left\{i\cdot a(a+1)+a\cdot j+a : 0\leq i\leq a^2-1, 0\leq j\leq a\right\}\setminus\{N+1\}, \text{ and}$$
$$\overline{S_a}(N)=\left\{i\cdot a(a+1)+a\cdot j+k : 0\leq i\leq a^2-1, 0\leq j\leq a, 1\leq k\leq a-1\right\}.$$

Let $\overline{S_a}(N)=R_\ell\cup B_\ell$ be a partition defined by

$$R_\ell=\left\{i\cdot a(a+1)+a\cdot j+k : 0\leq i\leq a^2-1, 0\leq j\leq a, 2\lfloor j/2\rfloor+1\leq k\leq a-1\right\}, \text{ and}$$
$$B_\ell=\left\{i\cdot a(a+1)+a\cdot j+k : 0\leq i\leq a^2-1, 0\leq j\leq a, 1\leq k\leq 2\lfloor j/2\rfloor\right\}.$$

Now, consider the set

$$S_{a^2}(N)=\left\{i\cdot a(a+1)+a\cdot j+a : 0\leq i\leq a^2-1, 0\leq j\leq a, a \mid (i+j+1)\right\}\setminus\{N+1\}.$$

Let $S_{a^2}(N)=R_r\cup B_r$ be a partition defined by

$$B_r=\bigcup_{i=1}^{a-1}\left\{a^4+ia^2\right\}\cup\bigcup_{i=1}^{a-1}\left\{ia^3+ja^2 : 0\leq j\leq i-1\right\}\cup\bigcup_{i=(a+1)/2}^{a-1}\left\{ia^3+ia^2\right\}, \text{ and}$$

$$R_r=\left\{a^4\right\}\cup\bigcup_{i=0}^{a-1}\left\{ia^3+ja^2 : i+1\leq j\leq a-1\right\}\cup\bigcup_{i=1}^{(a-1)/2}\left\{ia^3+ia^2\right\}.$$

Our colouring $\delta\colon [1, N] \to [0, 1, 2]$ will be defined by

$$\delta(x) = \begin{cases} 2 & \text{if } x \in B_\ell \cup B_r, \\ 1 & \text{if } x \in R_\ell \cup R_r, \\ 0 & \text{if } x \in S_a(N) \setminus S_{a^2}(N). \end{cases}$$

Proof of Theorem 1.4. A written proof of Theorem 1.4 is provided in Appendix C. The symbolic correctness of the partition and the verification of the cases can be found in the AutoCase repository. □

4. Conclusion

In this work, we explored the computation of 3-colour Rado numbers for certain 3-term linear equations. Our SAT-based approach enabled us to extend the known values of $R_3(\mathcal{E}(3, 0; a, b, b))$ and $R_3(\mathcal{E}(3, 0; a, a, b))$, computing previously unknown Rado numbers and formulating new conjectures based on observed patterns. We proved the lower bounds $R_3(\mathcal{E}(3, 0; a, b, b)) \geq a^3 + a^2 + (2b+1)a + 1$ and $R_3(\mathcal{E}(3, 0; a, a, a+1)) \geq a^3(a+1)$ for certain ranges of a and b. The case-based proofs were automatically verified using a combination of symbolic computation and SMT solvers in a tool we call AutoCase.

We did not attempt to compute Rado numbers involving more than three colours due to the rapid growth in the SAT instances, and for simplicity we focused on linear equations with a constant term of zero. In principle, our methods should be applicable to nonhomogeneous linear equations and to Rado numbers involving four or more colours, though they may require more computational resources or more advanced combinatorial optimization techniques. We leave such cases for future work.

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A. Proof of Lemma 2.1

In this section we assume a and b are positive coprime integers. Lemma 2.1 then says

$$R_1(\mathcal{E}(3,0;a,b,b)) = \max(a+1,b) \quad \text{and} \quad R_1(\mathcal{E}(3,0;a,a,b)) = \begin{cases} 2a & \text{if } b=1, \\ \max(a,\lceil b/2 \rceil) & \text{if } b>1. \end{cases} \tag{1}$$

For the first equation of (1), we want to find the solution (x,y,z) of $ax+by=bz$ that minimizes $\max(x,y,z)$; then $R_1(\mathcal{E}(3,0;a,b,b)) = \max(x,y,z)$. Since a and b are coprime, x must be divisible by b, so $(b,1,a+1)$ is the solution of $ax+by=bz$ with smallest possible x and y values. Since $z=(ax+by)/b$, the solution with the smallest possible x and y will also minimize z. Thus, the smallest possible value of $\max(x,y,z)$ is $\max(b,1,a+1) = \max(a+1,b)$.

For the second equation of (1), we want to find the solution (x,y,z) of $ax+ay=bz$ that minimizes $\max(x,y,z)$; then $R_1(\mathcal{E}(3,0;a,a,b)) = \max(x,y,z)$. Since a and b are coprime, z must be divisible by a, so say $z=ak$, giving $x+y=bk$. Then $\max(x,y,z)$ is minimized when x and y are as close to equal as possible (i.e., $x \approx y \approx bk/2$).

If b is even, $\max(x,y,z)$ is minimized when $k=1$ and $x=y=b/2$, giving $\max(x,y,z) = \max(a,b/2)$, so $R_1(\mathcal{E}(3,0;a,a,b)) = \max(a,b/2)$ when b is even. If $b>1$ is odd, then $\max(x,y,z)$ is minimized when $k=1$ and $\{x,y\} = \{\lfloor b/2 \rfloor, \lceil b/2 \rceil\}$, so $R_1(\mathcal{E}(3,0;a,a,b)) = \max(a,\lceil b/2 \rceil)$ when $b>1$ is odd. If $b=1$, then $\max(x,y,z)$ is minimized when $k=2$ and $x=y=1$ (note $x+y=k$ has no solutions in positive integers when $k=1$) and then $R_1(\mathcal{E}(3,0;a,a,b)) = \max(1,1,2a) = 2a$.

B. Written proof of Theorem 1.3

Suppose there is a solution to $ax+by=bz$ monochromatic in colour 0 (Dark Gray). Since $\gcd(a,b)=1$ and there is no multiple of b in D_1, the variable x must be in D_2. Taking $x=ba^2+ba+b$ and $y=\min(D_1)=ba+1$ as smallest possible values, we get $z=a^3+a^2+(b+1)a+1$, but $\delta(z)=1$ (Red) for $z \geq a^3+a^2+(b+1)a+1$, a contradiction.

Suppose there is a solution to $ax+by=bz$ monochromatic in colour 1 (Red). Since R_1 does not contain a multiple of b, variable x cannot be in R_1. Also, if $x,y \in R_3$, then clearly $z > a^3+a^2+(2b+1)a$. So, x and y both cannot be in R_3. Now we consider the following cases:

(a) $x \in R_2$ and $y \in R_1$: We have $z = \frac{ax}{b} + y = baw + y$ for $1 \leq w \leq a$. Since, $y \not\equiv 0 \pmod{b}$, we have $baw+y \not\equiv 0 \pmod{b}$ which implies $baw+y \not\equiv 0 \pmod{b^2}$. Hence $baw+y \notin R_2$. If $baw+y \in R_1$, then since $w \geq 1$ and $y \geq 1$, we have $baw+y > ba = \max(R_1)$, a contradiction.

(b) $x \in R_2$ and $y \in R_2$: We have $z \notin R_3$ since

$$\begin{aligned} z &\leq a(b^2a)/b + b^2a = ba^2 + b^2a \\ &\leq a^3+a^2+(b+1)a < \min(R_3) \end{aligned}$$

when $a^2+a+b > b^2+ba$. Suppose $z \in R_2$. In that case, $ax_1+bx_2=bx_3$ with $1 \leq x_1,x_2,x_3 \leq ab^2$ implies $ak_1+bk_2=bk_3$ with $1 \leq k_1,k_2,k_3 \leq a$. Then, $k_1=(b/a)(k_3-k_2)$ has no integer solution since $\gcd(a,b)=1$ and $0 \leq k_3-k_2 < a$. Hence, $\delta(z) \neq 1$.

(c) $x \in R_3$ and $y \in R_1$: Consider the condition, with $a^3+a^2+(b+1)a+c$ being the smallest integer in R_3 which is divisible by b. Then, we have $a^2+a+b > ba+b^2$ which can be re-written as $\frac{a}{b}(a^3+a^2+(b+1)a+c)+1 > \frac{a}{b}(ba^2+ab^2+a+c)+1$, which implies

$$z > (a^3+a^2b+a^2/b+ac/b) > \max(R_3).$$

Hence, $\delta(z) \neq 1$, a contradiction.

(d) $x \in R_3$ and $y \in R_2$: As in the previous case, with $a^3 + a^2 + (b+1)a + c$ being the smallest integer in R_3 which is divisible by b, and b^2 being the smallest integer in R_2, we get $z > \max(R_3)$ implying $z \notin R_3$ implying $\delta(z) \neq 1$, a contradiction.

Suppose there is a solution to $ax + by = bz$ monochromatic in colour 2 (Blue). Then we have the following cases:

(i) If $x, y \in B_2$, then

$$\begin{aligned} z = \frac{ax}{b} + y &\geq \frac{a(b^2a + b)}{b} + (b^2a + b) \\ &= ba^2 + b^2a + a + b > \max(B_2), \end{aligned}$$

which results a contradiction since integers beyond $\max(B_2)$ are coloured in either colour 0 or in colour 1.

(ii) If $x, y \in B_1$, then in the equation $z = \frac{ax}{b} + y$, we have $\frac{x}{b} \equiv 1, 2, \dots, b-1 \pmod{b}$. Since $\gcd(a, b) = 1$, we have $\frac{ax}{b} \not\equiv 0 \pmod{b}$. Hence, $z = \frac{ax}{b} + y \not\equiv b, 2b, \dots, b(b-1) \pmod{b^2}$ and $z \not\equiv 0 \pmod{b}$ implying $\delta(z) \neq 2$, a contradiction.

(iii) If $x \in B_1$ and $y \in B_2$, then as in the previous case, we have $\frac{ax}{b} \not\equiv 0 \pmod{b}$. Since $y \equiv 0 \pmod{b}$, we have $\frac{ax}{b} + y \not\equiv 0 \pmod{b}$, that is, $z \notin B_2$. Hence, $\delta(z) \neq 2$, a contradiction.

(iv) If $x \in B_2$ and $y \in B_1$, then $z = \frac{ax}{b} + y \geq a(b^2a + b)/b + b = a^2b + a + b > \max(B_1)$. Also, since $z \not\equiv 0 \pmod{b}$, we have $z \notin B_2$. Hence, $\delta(z) \neq 2$, a contradiction.

Hence,

$$R_3(\mathcal{E}(3, 0; a, b, b)) \geq a^3 + a^2 + (2b+1)a + 1$$

if $a^2 + a + b > b^2 + ba$.

C. Written proof of Theorem 1.4

C.1. Integrality properties

Lemma C.1. *Let a be a positive integer. There is no integer solution x_1, x_2, x_3 to the equation $ax_1 + ax_2 = (a+1)x_3$ if any of the following is true:*
a. $x_1, x_2, x_3 \not\equiv 0 \pmod{a}$;
b. $x_1, x_2, x_3 \equiv 0 \pmod{a}$ *and* $x_1, x_2, x_3 \not\equiv 0 \pmod{a^2}$;
c. $x_1, x_2 \equiv 0 \pmod{a^2}$ *and* $x_3 \not\equiv 0 \pmod{a}$;
d. $x_1, x_3 \equiv 0 \pmod{a^2}$ *and* $x_2 \not\equiv 0 \pmod{a}$;
e. $x_2, x_3 \equiv 0 \pmod{a^2}$ *and* $x_1 \not\equiv 0 \pmod{a}$;
f. $x_1 \equiv 0 \pmod{a^2}$ *and* $x_2, x_3 \not\equiv 0 \pmod{a}$;
g. $x_2 \equiv 0 \pmod{a^2}$ *and* $x_1, x_3 \not\equiv 0 \pmod{a}$;

Proof. In each of the cases, assume that an integer solution x_1, x_2, x_3 exists to $ax_1 + ax_2 = (a+1)x_3$.
a. Simplifying the equation, we get $x_1 + x_2 = x_3 + x_3/a$, which due to the integrality assumption, implies $a|x_3$, that is, $x_3 \equiv 0 \pmod{a}$, a contradiction.
b. For $i = 1, 2, 3$, since $x_i \equiv 0 \pmod{a}$ and $x_i \not\equiv 0 \pmod{a^2}$, let $x_i = a \cdot k_i$ such that $k_i \not\equiv 0 \pmod{a}$. Upon substitution, we obtain $a^2 \cdot (k_1 + k_2 - k_3) = a \cdot k_3$ which implies $k_3 \equiv 0 \pmod{a}$, a contradiction.
c. Let $x_1 = a^2k_1$ and $x_2 = a^2k_2$. Then after substitution and simplification, we obtain $a^2k_1 + a^2k_2 = x_3 + x_3/a$, which due to the integrality assumption, implies $a|x_3$, that is, $x_3 \equiv 0 \pmod{a}$, a contradiction.

d. Let $x_1 = a^2k_1$ and $x_3 = a^2k_3$. Then after substitution and simplification, we obtain $ak_1 + x_2/a = ak_3 + k_3$, which due to the integrality assumption, implies $a|x_2$, that is, $x_2 \equiv 0 \pmod{a}$, a contradiction.
e. Let $x_2 = a^2k_2$ and $x_3 = a^2k_3$. Then after substitution and simplification, we obtain $x_1/a + ak_2 = ak_3 + k_3$, which due to the integrality assumption, implies $a|x_1$, that is, $x_1 \equiv 0 \pmod{a}$, a contradiction.
f. Let $x_1 = a^2k_1$. Then after substitution and simplification, we obtain $a^2k_1 + x_2 = x_3 + x_3/a$, which due to the integrality assumption, implies $a|x_3$, that is, $x_3 \equiv 0 \pmod{a}$, a contradiction.
g. Let $x_2 = a^2k_2$. Then after substitution and simplification, we obtain $x_1 + a^2k_2 = x_3 + x_3/a$, which due to the integrality assumption, implies $a|x_3$, that is, $x_3 \equiv 0 \pmod{a}$, a contradiction. □

C.2. Proof of good colouring

Lemma C.2. *For odd positive integer $a \geq 7$, there exists no monochromatic solution to $ax_1 + ax_2 = (a+1)x_3$ in colour 0.*

Proof. If $x \in S_a(N) \setminus \overline{S_{a^2}}(N)$, then by Lemma C.1(b), there exists no solution to $ax_1 + ax_2 = (a+1)x_3$. Hence there is no solution to $ax_1 + ax_2 = (a+1)x_3$ monochromatic in colour 0. □

Lemma C.3. *There exists no monochromatic solution to $ax_1 + ax_2 = (a+1)x_3$ if $x_1, x_2, x_3 \in R_r$.*

Proof. Given $x_1, x_2 \in R_r$, assume that $x_3 \in R_r$ where $x_3 = (ax_1 + ax_2)/(a+1)$. Let $x_1 = k_1a^2$ and $x_2 = k_2a^2$ for positive integers k_1 and k_2. Substituting for x_3, we get the numerator to be $a^3(k_1 + k_2)$. Since a^3 shares no factor with $a+1$, if $a+1$ does not divide $k_1 + k_2$, then $x_3 \notin \mathbb{Z}$, a contradiction. Assume that $k_1 + k_2 = m(a+1)$ for some positive integer m.

Assume that $k_1 + k_2 = m(a+1)$ for some positive integer m. Then, $z = a^3m$, but for $1 \leq m \leq a-1$, by definition of the sets B_r and R_r, we have $a^3m \in B_r$ and since $B_r \cap R_r = \emptyset$, we have $a^3m \notin R_r$. If $z = a^4$, then $m = a$ implying $k_1 + k_2 = a(a+1)$ which is impossible since $k_1 + k_2 \leq ai + j \leq a^2 - 1 < a(a+1)$ given $0 \leq i \leq a-1, i+1 \leq j \leq a-1$. □

Lemma C.4. *For odd positive integer $a \geq 7$, there exists no monochromatic solution to $ax_1 + ax_2 = (a+1)x_3$ if $x_1, x_2 \in R_\ell$ and $x_3 \in R_r$.*

Proof. Since $x_1, x_2 \in R_\ell$, let $x_1 = i_1a(a+1) + r_1$ and $x_2 = i_2a(a+1) + r_2$ with $0 \leq i_1, i_2 \leq a^2 - 1$, $1 \leq r_1, r_2 < a(a+1)$. Also, let $x_3 = a^2k_3$. Therefore,

$$k_3 = (i_1 + i_2) + \frac{r_1 + r_2}{a(a+1)}.$$

If $r_1 + r_2 = a(a+1)$, then by construction of B_ℓ and R_ℓ, the integers x_1 and x_2 both not in the same set, a contradiction. □

Lemma C.5. *For odd positive integer $a \geq 7$, there exists no monochromatic solution to $ax_1 + ax_2 = (a+1)x_3$ in colour 1.*

Proof. Let $R_r = R_1 \cup R_2 \cup R_3$ with $R_1 = \{a^4\}$, $R_2 = \bigcup_{i=1}^{a-1} \{ia^3 + ja^2 : i+1 \leq j \leq a-1\}$, and $R_3 = \bigcup_{i=1}^{(a-1)/2} \{ia^3 + ia^2\}$. There are $4^3 = 64$ ways to select an integer for the variables x_1, x_2, x_3 from the 4 sets R_ℓ, R_1, R_2, R_3. Most of these cases can be analyzed directly using Lemma C.1 as follows:
- $x_1, x_2, x_3 \in R_\ell$: No monochromatic solution by Lemma C.1 (a) covering 1 case.
- $x_1 \in R_\ell, x_2, x_3 \in R_r$: No monochromatic solution by Lemma C.1 (e) covering 9 cases.
- $x_2 \in R_\ell, x_1, x_3 \in R_r$: No monochromatic solution by Lemma C.1 (d) covering 9 cases.
- $x_3 \in R_\ell, x_1, x_2 \in R_r$: No monochromatic solution by Lemma C.1 (c) covering 9 cases.
- $x_2, x_3 \in R_\ell, x_1 \in R_r$: No monochromatic solution by Lemma C.1 (f) covering 3 cases.
- $x_1, x_3 \in R_\ell, x_2 \in R_r$: No monochromatic solution by Lemma C.1 (g) covering 3 cases.
- $x_1, x_2 \in R_\ell, x_3 \in R_r$: No monochromatic solution by Lemma C.4 covering 3 cases.

- $x_1, x_2, x_3 \in R_r$: No monochromatic solution by Lemma C.3 covering 27 cases.

Hence, there exists no monochromatic solution to $ax_1 + ax_2 = (a+1)x_3$ in colour 1. $\square$

Lemma C.6. *For an odd positive integer $a \geq 7$, the set B_r contains no solution to $\mathcal{E}(3, 0; a, a, a+1)$.*

Proof. Given $x_1, x_2 \in B_r$, assume that $x_3 = \frac{ax_1+ax_2}{a+1}$ and $x_3 \in B_r$. Let $x_1 = k_1a^2$ and $x_2 = k_2a^2$ for positive integers k_1 and k_2. Substituting for x_3, we get the numerator to be $a^3(k_1+k_2)$. Since a^3 shares no factor with $a+1$, if $a+1$ does not divide k_1+k_2, then $z \notin \mathbb{Z}$, a contradiction. Assume that $k_1+k_2 = m(a+1)$ for some positive integer m. Then, $z = a^3m$ and assume $z \in B_r$. For $1 \leq i \leq a-1$, $a^4 + ia^2$ is not an integer multiple of a^3. Since $\min(B_r) = a^3$, we have $k_1 + k_2 \geq 2a$ and the smallest integer value of m is obtained when $k_1 + k_2 = (a+1)a$. This implies $z \geq a^4$, but $\max(B_2 \cup B_3) < a^4$. Hence, $z \notin B_r$.

Therefore, the set B_r contains no solution to $ax_1 + ax_2 = (a+1)x_3$. $\square$

Lemma C.7. *For odd positive integer $a \geq 7$, there exists no monochromatic solution to $ax_1 + ax_2 = (a+1)x_3$ if $x_1, x_2 \in B_\ell$ and $x_3 \in B_r$.*

Proof. Similar to the proof of Lemma C.4. $\square$

Lemma C.8. *For odd positive integer $a \geq 7$, there exists no monochromatic solution to $ax_1 + ax_2 = (a+1)x_3$ in colour 2.*

Proof. Let $B_r = B_1 \cup B_2 \cup B_3$ with partition blocks $B_1 = \bigcup_{i=1}^{a-1}\left\{a^4 + ia^2\right\}$, $B_2 = \bigcup_{i=1}^{a-1}\left\{ia^3 + ja^2 : 0 \leq j \leq i-1\right\}$, and $B_3 = \bigcup_{i=(a+1)/2}^{a-1}\left\{ia^3 + ia^2\right\}$. There are $4^3 = 64$ ways to select an integer for the variables x_1, x_2, x_3 from the 4 sets B_ℓ, B_1, B_2, B_3. Most of these cases can be analyzed directly using Lemma C.1 as follows:

- $x_1, x_2, x_3 \in B_\ell$: No monochromatic solution by Lemma C.1 (a) covering 1 case.
- $x_1 \in B_\ell, x_2, x_3 \in B_r$: No monochromatic solution by Lemma C.1 (e) covering 9 cases.
- $x_2 \in B_\ell, x_1, x_3 \in B_r$: No monochromatic solution by Lemma C.1 (d) covering 9 cases.
- $x_3 \in B_\ell, x_1, x_2 \in B_r$: No monochromatic solution by Lemma C.1 (c) covering 9 cases.
- $x_2, x_3 \in B_\ell, x_1 \in B_r$: No monochromatic solution by Lemma C.1 (f) covering 3 cases.
- $x_1, x_3 \in B_\ell, x_2 \in B_r$: No monochromatic solution by Lemma C.1 (g) covering 3 cases.
- $x_1, x_2 \in B_\ell, x_3 \in B_r$: No monochromatic solution by Lemma C.7 covering 3 cases.
- $x_1, x_2, x_3 \in B_r$: No monochromatic solution by Lemma C.6 covering 27 cases.

Hence, there exists no monochromatic solution to $ax_1 + ax_2 = (a+1)x_3$ in colour 2. $\square$

D. SMT Instance via the Z3 Python API for Example 3.1

```
from z3 import *

# Declare symbolic variables
a = Int('a')
x1, x2, x3 = Ints('x_1 x_2 x_3')
k1, k2 = Ints('k_1 k_2')
i, j, k = Ints('i j k')

# Create the solver
s = Solver()

# Global assumptions
s.add(a >= 1)

# Equation constraint
```

```
s.add(a * x1 + a * x2 == (a + 1) * x3)

# Divisibility: a | x1
s.add(And(k1>0, x1 == a * k1))

# Non-divisibility: not(a^2 | x2)
s.add(ForAll([k2], And(k2>0, x2 != a**2 * k2)))

# Format expression for x3
s.add(x3 == a*(a + 1)*i + a*j + k)

# Symbolic format bounds
s.add(i >= 0, i < a**2)
s.add(j >= 0, j <= a)
s.add(k >= 2*(j // 2) + 1, k <= a - 1)

# Bounds for x1, x2, x3
s.add(x1 >= 1, x1 <= a**2)
s.add(x2 >= 1, x2 <= a**2)
s.add(x3 >= 1, x3 <= a**2)
```