Searching for projective planes with **computer algebra** and **SAT solvers**

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July 19, 2019

SAT:

Boolean satisfiability problem

SAT solvers: Clever brute force

Effectiveness of SAT solvers

Many problems that have nothing to do with logic can be effectively solved by reducing them to Boolean logic and using a SAT solver.

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Examples

- Hardware and software verification
- Scheduling subject to constraints
- Finding or disproving the existence of combinatorial objects

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- Finding or disproving the existence of combinatorial objects

Limitations

Lack of expressiveness, and SAT solvers perform poorly on highly symmetric problems.

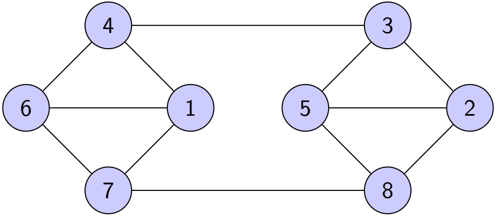
CAS:

Computer algebra system

Symbolic mathematical computing

Example

What is the automorphism group of this graph?



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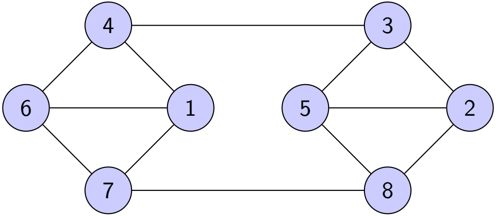

MAPLE returns

$$\langle (2,5),\quad (3,8)(4,7),\quad (1,2)(3,4)(5,6)(7,8) \rangle.$$

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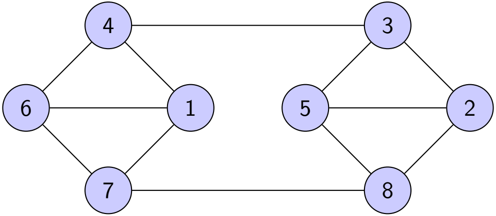


MAPLE returns

$$\langle (2,5),\quad (3,8)(4,7),\quad (1,2)(3,4)(5,6)(7,8) \rangle.$$

Limitations

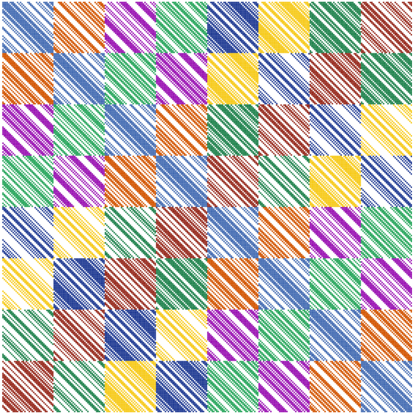
CASs are not optimized to do large (i.e., exponential) searches.

SAT + CAS

Brute force + Knowledge

MathCheck

Our SAT+CAS system MathCheck has constructed over 100,000 various combinatorial objects. For example, this $\{\pm 1\}$-matrix with pairwise orthogonal rows:



Results first shown by MathCheck

- Found the smallest counterexample of the Williamson conjecture.
- Verified the even Williamson conjecture up to order 70.
- Found three new counterexamples to the good matrix conjecture.
- Verified the best matrix conjecture up to order seven.
- Verified the Ruskey–Savage conjecture up to order five.
- Verified the Norine conjecture up to order six.

Details available at:

`uwaterloo.ca/mathcheck`

Projective planes

A projective plane is a set of points and lines and a relation between points and lines such that:

- There is a unique line between any two points.
- Any two lines meet at a unique point.

Projective planes of order n

A finite projective plane is a collection of $n^2 + n + 1$ lines and $n^2 + n + 1$ points such that:

- There are $n + 1$ points on each line.
- There are $n + 1$ lines through each point.

Incidence matrix representation

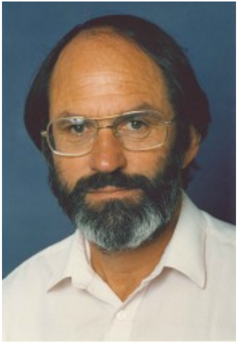
Projective plane of order 2:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- $\{0,1\}$-matrix of size 7×7.
- Each row (representing lines) contains exactly three 1s.
- Each column (representing points) contains exactly three 1s.

For what orders do projective planes exist?

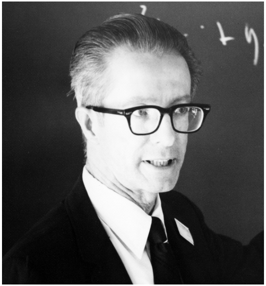
...every known plane has prime power order ... [and] has been constructed in one way or another from a finite field...



Peter Lorimer
The Construction of Finite Projective Planes
1981

The Bruck–Ryser theorem

If n is the order of a projective plane and $n \equiv 1, 2 \pmod 4$ then n is the sum of two squares.



Projective planes of small orders

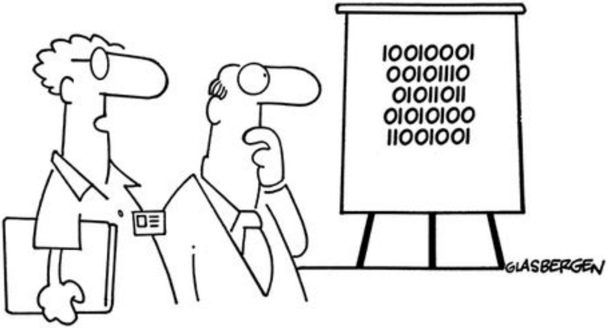
2	3	4	5	6	7	8	9	10	11	12	13	14	15
✓	✓	✓	✓	✗	✓	✓	✓	?	✓	?	✓	✗	?

✓ Finite field construction
✗ Bruck–Ryser theorem

The first critical value of n is $n = 10$. A thorough investigation of this case is currently beyond the facilities of computing machines.

Marshall Hall Jr.
Finite Projective Planes
1955

Enter coding theory



"We've devised a new security encryption code. Each digit is printed upside down."

Codewords

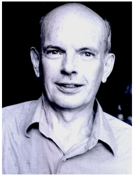
A *codeword* generated by a projective plane is a vector in the row space of its incidence matrix (over $F_2 = \{0, 1\}$).

The *weight* of a codeword is the number of 1s it contains.

A search for weight 15 codewords

In 1970, MacWilliams, Sloane, and Thompson showed that a projective plane of order ten must generate weight 15, 16, or 19 codewords.

Furthermore, they used three hours of computing on a mainframe computer to show that codewords of weight 15 do not exist.



Other searches

We know of three other searches with code we could run:

- [Dominique Roy, 2005] Implementation in C, runs in 78 minutes.
- [Casiello, Indaco, and Nagy, 2010] Implementation in GAP, runs in 7 minutes.
- [Xander Perrott, 2016] Implementation in MATHEMATICA, runs in 55 minutes.

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Our result

We verified the search using a SAT+CAS method in seconds.

Searches for weight 16 codewords

In 1974, Carter performed a partial search for weight 16 codewords using approximately 140 hours on a mainframe computer.

Searches for weight 16 codewords

In 1986, Lam, Thiel, and Swiercz completed the weight 16 search using about 1,900 hours of computing on a VAX-11/780.



Searches for codewords of weight 19

In 1989, Lam, Thiel, and Swiercz used about 19,200 hours on a VAX-11/780 and 2,000 hours on a CRAY-1A supercomputer run by the Institute for Defense Analyses to show that no weight 19 codewords exist.



I'm sorry, but that's the way it goes. The order 12 case is open, by the way, but a computer attack along the same lines would take ten thousand million times as long.

Ian Stewart
Another Fine Math
You've Got Me Into. . .
1992

Using SAT solvers for combinatorial search

Surprisingly, SAT solving is getting so strong that indeed [SAT solvers seem] today the best solution in most cases.



Marijn Heule,
Oliver Kullmann,
Victor Marek
Solving Very Hard Problems: Cube-and-Conquer, a Hybrid SAT Solving Method
2017

If a weight 15 codeword exists, MacWilliams, Sloane, and Thompson showed that the first 21 rows (up to equivalence) of the incidence matrix of a projective plane of order ten are exactly:

```
111110000000000000000000000000000000000000000000000000000000000000000000000111111000000000000000000000000000000
100001111000000000000000000000000000000000000000000000000000000000000000000000000111111000000000000000000000000
010001000111000000000000000000000000000000000000000000000000000000000000000000000000000111111000000000000000000
001000100100110000000000000000000000000000000000000000000000000000000000000000000000000000000111111000000000000
000100010010101000000000000000000000000000000000000000000000000000000000000000000000000000000000000111111000000
000010001001011000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000111111
100000000100001000000000000000000000000000000110000110000110000110000000000000000000000000000000000000000000000
100000000010010000000000000000000000000001100000000000001100001100000001100000000000000000000000000000000000000
100000000000110000000000000000000000000000110000110000000000000000110000110000000000000000000000000000000000000
010000100000001000000000000000001110100000000000000000000000101000101000000000000000000000000000000000000000000
010000010000010000000000110000000000011000000000000000000101000000000000101000000000000000000000000000000000000
010000000100010000000011000011000000000000000000000000000000000101000101000000000000000000000000000000000000000
001001000000001000000000000011000000110000000000101000011000000000000000000000000000000000000000000000000000000
001000010001000110000000000000000011000000000101000000000000000000000011000000000000000000000000000000000000000
001000001010000001010000000000101000000000000000000101000000000000000000011000000000000000000000000000000000000
000101000000010000000101000000000110000000101000000011000000000000000000000000000000000000000000000000000000000
000100100001000001001000000000000000101101000000000000000000000011000000000000000000000000000000000000000000000
000100001100000100100000011000000000000000000000000000101000000000011000000000000000000000000000000000000000000
000011000000100000000000101000110000000011000011000000000000000000000000000000000000000000000000000000000000000
000010100010000010100000000101000000000000011000000000000011000000000000000000000000000000000000000000000000000
000010010100000000011011000000000000000000000000011000000000011000000000000000000000000000000000000000000000000
```

The next 24 rows are of this form, where blanks are unknown entries:

```
100000000000000                        00 00 00 00 00 00 00 00 00 00 00 00 000000000000
100000000000000                        00 00 00 00 00 00 00 00 00 00 00 00 000000000000
100000000000000                        00 00 00 00 00 00 00 00 00 00 00 00 000000000000
100000000000000                        00 00 00 00 00 00 00 00 00 00 00 00 000000000000
100000000000000                        00 00 00 00 00 00 00 00 00 00 00 00 000000000000
100000000000000                        00 00 00 00 00 00 00 00 00 00 00 00 000000000000
0000000001000000  000 00 00                  00  0000 0 000  0000  00                  000000000000
0000000001000000  000 00 00                  00  0000 0 000  0000  00                  000000000000
0000000001000000  000 00 00                  00  0000 0 000  0000  00                  000000000000
0000000001000000  000 00 00                  00  0000 0 000  0000  00                  000000000000
0000000001000000  000 00 00                  00  0000 0 000  0000  00                  000000000000
0000000001000000  000 00 00                  00  0000 0 000  0000  00                  000000000000
000000000000001             00 000 000       00 0 000  0000 0 000 0 0                              000000000000
000000000000001             00 000 000       00 0 000  0000 0 000 0 0                              000000000000
000000000000001             00 000 000       00 0 000  0000 0 000 0 0                              000000000000
000000000000001             00 000 000       00 0 000  0000 0 000 0 0                              000000000000
000000000000001             00 000 000       00 0 000  0000 0 000 0 0                              000000000000
000000000000001             00 000 000       00 0 000  0000 0 000 0 0                              000000000000
000000000010000 0000       0 00 0      00  00      0 000  0000       00  00            000000      000000
000000000010000 0000       0 00 0      00  00      0 000  0000       00  00            000000      000000
000000000010000 0000       0 00 0      00  00      0 000  0000       00  00            000000      000000
000000000010000 0000       0 00 0      00  00      0 000  0000       00  00            000000      000000
000000000010000 0000       0 00 0      00  00      0 000  0000       00  00            000000      000000
000000000010000 0000       0 00 0      00  00      0 000  0000       00  00            000000      000000
```

SAT encoding

Consider lines 1 and 28:

```
11111000000000000000...000000000000000000001111110000...
0000000001000000   00...00   0000   00                    ...
```

SAT encoding

Consider lines 1 and 28:

```
11111000000000000000...000000000000000000111111000...
0000000001000000  00...00  0000  00                ...
```

There must be some point that is on both of these lines.

SAT encoding

Consider lines 1 and 28:

```
11111000000000000000...000000000000000000111111000...
0000000001000000  00...00  0000  00      ******   ...
```

There must be some point that is on both of these lines.

A 1 must appear here.

SAT encoding

Consider lines 1 and 28:

```
11111000000000000000...000000000000000000111111000...
0000000001000000   00...00   0000   00        abcdef    ...
```

There must be some point that is on both of these lines.

A 1 must appear here.

In Boolean logic:

$$a \vee b \vee c \vee d \vee e \vee f$$

SAT encoding

Consider lines 22 and 40:

```
100000000000000            ...
0000000000100000 0000      ...
```

SAT encoding

Consider lines 22 and 40:

```
1000000000000000          ...
0000000000010000 0000     ...
```

Exactly one point must appear on both of these two lines.

SAT encoding

Consider lines 22 and 40:

```
1000000000000000*     *      ...
0000000000010000*0000*      ...
```

Exactly one point must appear on both of these two lines.

These cannot all simultaneously be 1.

SAT encoding

Consider lines 22 and 40:

```
1000000000000000a    b     ...
0000000000100000c0000d     ...
```

Exactly one point must appear on both of these two lines.

These cannot all simultaneously be 1.

In Boolean logic:

$$\neg a \vee \neg b \vee \neg c \vee \neg d$$

Solving the SAT instance

Up to 27 rows, the SAT instance has about 150 unknown variables, 1000 clauses, and over 10^{18} solutions.

However, many columns are rows are identical and permuting them produces other equivalent solutions.

Symmetry breaking

Using appropriate row/column permutations, we can assume the first 27 rows are:

```
111110000000000000000000000000000000000000000000000000000000000000000000000111111000000000000000000000000000000
100001111000000000000000000000000000000000000000000000000000000000000000000000000111111000000000000000000000000
010001000111000000000000000000000000000000000000000000000000000000000000000000000000000111111000000000000000000
001000100100110000000000000000000000000000000000000000000000000000000000000000000000000000000111111000000000000
000100010010101000000000000000000000000000000000000000000000000000000000000000000000000000000000000111111000000
000010001001011000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000111111
100000000100001000000000000000000000000000000110000110000110000110000000000000000000000000000000000000000000000
100000000010010000000000000000000000000110000000000000110000110000000011000000000000000000000000000000000000000
100000000001100000000000000000000000000000110000110000000000000000110000110000000000000000000000000000000000000
010000100000001111111000000000000000000000000000000000000000100000100000000000000000000000000000000000000000000
010000010000010000000000110000000000011000000000000000000101000000000000101000000000000000000000000000000000000
010000001000100000000110000110000000000000000000000000000000000101000101000000000000000000000000000000000000000
001001000000001000000000000000110000001100000000101000011000000000000000000000000000000000000000000000000000000
001000010001000000100000000000011010000000000101000000000000000000000011000000000000000000000000000000000000000
001000001010000010000000000000100100000000000000000001010000000000100000000011000000000000000000000000000000000
000101000000010001000101000000000000100000000010100000000011000000000000000000000000000000000000000000000000000
000100100001000000000000000000000001001011010000000000000000000000001100100000000000000000000000000000000000000
000100001100000000001000011000010001000000000000000000000010100000000000010000000000000000000000000000000000000
000011000000100100000000101000100000000011000011000000000000000000000000000000000000000000000000000000000000000
000010100010000000000000000101001001000000011000000000000000001100000000000000000000000000000000000000000000000
000010010100000000010011000000000000000000000000011000000000011000001000000000000000000000000000000000000000000
100000000000000                        00 00 00 00 00 00 00 00 00 00 00 00 000000000000
100000000000000                        00 00 00 00 00 00 00 00 00 00 00 00 000000000000
100000000000000                        00 00 00 00 00 00 00 00 00 00 00 00 000000000000
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100000000100001000000000000000000000000000000110000110000110000110000000000000000000000000000000000000000000000
100000000010010000000000000000000000001100000000000000110000110000000110000000000000000000000000000000000000000
100000000001100000000000000000000000000000110000110000000000000000110000110000000000000000000000000000000000000
010000100000001111111000000000000000000000000000000000000000100000100000000000000000000000000000000000000000000
010000010000010000000000110000000000011000000000000000000101000000000000101000000000000000000000000000000000000
010000001000100000000110000110000000000000000000000000000000000101000101000000000000000000000000000000000000000
001001000000001000000000000011000000110000000001010000110000000000000000000000000000000000000000000000000000000
001000010001000000100000000000011010000000000101000000000000000000000011000000000000000000000000000000000000000
001000001010000010000000000000100100000000000000000101000000000100000000001100000000000000000000000000000000000
000101000000010001000101000000000010000000101000000011000000000000000000000000000000000000000000000000000000000
000100100001000000000000000000000100101101000000000000000000000011001000000000000000000000000000000000000000000
000100001100000000001000011000010001000000000000000000101000000000100000000000000000000000000000000000000000000
000011000000100100000000101000100000000011000011000000000000000000000000000000000000000000000000000000000000000
000010100010000000000000000101001001000000011000000000000011000000000000000000000000000000000000000000000000000
000010010100000000010011000000000000000000000000011000000000011000001000000000000000000000000000000000000000000
1000000000000001                       00 00 00 00 00 00 00 00 00 00 00 00 000000000000
100000000000000 1                      00 00 00 00 00 00 00 00 00 00 00 00 000000000000
100000000000000  1                     00 00 00 00 00 00 00 00 00 00 00 00 000000000000
100000000000000   1                    00 00 00 00 00 00 00 00 00 00 00 00 000000000000
100000000000000    1                   00 00 00 00 00 00 00 00 00 00 00 00 000000000000
100000000000000     1                  00 00 00 00 00 00 00 00 00 00 00 00 000000000000
```

These rows can be sorted using row permutations.

Symmetry breaking

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100000000100001000000000000000000000000000000110000011000001100000110000000000000000000000000000000000000000000
100000000010010000000000000000000000000110000000000000110000110000000110000000000000000000000000000000000000000
100000000001100000000000000000000000000000110000110000000000000000110000110000000000000000000000000000000000000
010000100000001111111000000000000000000000000000000000000000100000100000000000000000000000000000000000000000000
010000010000010000000000110000000000011000000000000000000101000000000000101000000000000000000000000000000000000
010000001000100000000110000110000000000000000000000000000000000101000101000000000000000000000000000000000000000
001001000000001000000000000011000000110000000000101000011000000000000000000000000000000000000000000000000000000
001000010001000000100000000000011010000000000101000000000000000000000011000000000000000000000000000000000000000
001000001010000010000000000000100100000000000000000101000000000010000000000110000000000000000000000000000000000
000101000000010001000101000000000000100000000000101000000000110000000000000000000000000000000000000000000000000
000100100001000000000000000000000100101101000000000000000000000000001100100000000000000000000000000000000000000
000100001100000000001000011000010001000000000000000000001010000000000001000000000000000000000000000000000000000
000011000000100100000000101000100000000011000011000000000000000000000000000000000000000000000000000000000000000
000010100010000000000000000101001001000000011000000000000000001100000000000000000000000000000000000000000000000
000010010100000000010011000000000000000000000000011000000000011000001000000000000000000000000000000000000000000
1000000000000001                       00 00 00 00 00 00 00 00 00 00 00 00 0000000000001     1     1     1
100000000000000 1                      00 00 00 00 00 00 00 00 00 00 00 00 000000000000 1     1     1     1
100000000000000  1                     00 00 00 00 00 00 00 00 00 00 00 00 000000000000  1     1     1     1
100000000000000   1                    00 00 00 00 00 00 00 00 00 00 00 00 000000000000   1     1     1     1
100000000000000    1                   00 00 00 00 00 00 00 00 00 00 00 00 000000000000    1     1     1     1
100000000000000     1                  00 00 00 00 00 00 00 00 00 00 00 00 000000000000     1     1     1     1
```

These columns can be sorted using column permutations.

Symmetry breaking

Using appropriate row/column permutations, we can assume the first 27 rows are:

```
111110000000000000000000000000000000000000000000000000000000000000000000000111111000000000000000000000000000000
100001111000000000000000000000000000000000000000000000000000000000000000000000000111111000000000000000000000000
010001000111000000000000000000000000000000000000000000000000000000000000000000000000000111111000000000000000000
001000100100110000000000000000000000000000000000000000000000000000000000000000000000000000000111111000000000000
000100010010101000000000000000000000000000000000000000000000000000000000000000000000000000000000000111111000000
000010001001011000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000111111
100000000100001000000000000000000000000000000110000110000110000110000000000000000000000000000000000000000000000
100000000010010000000000000000000000000110000000000000110000110000000110000000000000000000000000000000000000000
100000000001100000000000000000000000000000011000011000000000000000011000011000000000000000000000000000000000000
010000100000001111111000000000000000000000000000000000000000100000100000000000000000000000000000000000000000000
010000010000010000000000110000000000011000000000000000000101000000000000101000000000000000000000000000000000000
010000001000100000000110000110000000000000000000000000000000000101000101000000000000000000000000000000000000000
001001000000001000000000000011000000110000000000101000011000000000000000000000000000000000000000000000000000000
001000010001000000100000000000011010000000000101000000000000000000000011000000000000000000000000000000000000000
001000001010000010000000000000100100000000000000001010000000010000000000110000000000000000000000000000000000000
000101000000010001000101000000000010000000101000000011000000000000000000000000000000000000000000000000000000000
000100100001000000000000000000000100101101000000000000000000000011001000000000000000000000000000000000000000000
000100001100000000001000011000010001000000000000000000101000000000100000000000000000000000000000000000000000000
000011000000100100000000101000100000000011000011000000000000000000000000000000000000000000000000000000000000000
000010100010000000000000000101001001000000011000000000000011000000000000000000000000000000000000000000000000000
000010010100000000010011000000000000000000000000011000000000110000010000000000000000000000000000000000000000000
1000000000000001                       00 00 00 00 00 00 00 00 00 00 00 00 0000000000001     1     1     1
100000000000000 1                      00 00 00 00 00 00 00 00 00 00 00 00 000000000000 1     1     1     1
100000000000000  1                     00 00 00 00 00 00 00 00 00 00 00 00 000000000000  1     1     1     1
100000000000000   1                    00 00 00 00 00 00 00 00 00 00 00 00 000000000000   1     1     1     1
100000000000000    1                   00 00 00 00 00 00 00 00 00 00 00 00 000000000000    1     1     1     1
100000000000000     1                  00 00 00 00 00 00 00 00 00 00 00 00 000000000000     1     1     1     1
```

Now 42,496 solutions.

Solving the SAT instances

The instances with up to 42 rows can now be solved in seconds.
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The instance with 43 rows is unsatisfiable and requires about 7 minutes to solve.

Solving the SAT instances

The instances with up to 42 rows can now be solved in seconds. These instances are all satisfiable.

The instance with 43 rows is unsatisfiable and requires about 7 minutes to solve.

This verifies the search of MacWilliams–Sloane–Thompson, but we can do better. . .

Using CAS to speed up the search

CAS symmetry breaking

A CAS can be used to find symmetries of the partially filled incidence matrix.

There are 48 symmetries that fix the already assigned entries in the first 27 rows.

Isomorphism blocking

When the SAT solver finds a solution of the first 27 rows, we use the 48 symmetries to block all isomorphic solutions.

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It takes just 6.5 seconds to show that the SAT instances up to 43 rows generated by these 1,021 solutions are unsatisfiable.

This verifies MacWilliams–Sloane–Thompson's search in 9 seconds.

Weight 16 searches

In 1974, Carter spent ∼140 hours on a mainframe. We verified his searches in 7 hours.

In 1986, Lam, Thiel, and Swiercz spent ∼1,900 hours of computing on a VAX-11/780. We verified their searches in 124 hours.

Weight 16 searches

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Next steps

We are currently working on verifying the weight 19 searches to produce a fully independent verification of the order ten search.