Computational bounds for book Ramsey numbers

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Introduction

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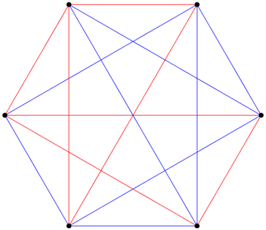
Every party of 6 people has a group of 3 mutual friends or 3 mutual strangers:



Introduction

Theorem (Ramsey, 1930)

Given positive integers r and s, there exists an integer n such that every red/blue edge coloring of the complete graph K_n contains either a red K_r or a blue K_s.

The Ramsey number $R(r, s)$ is the *smallest* such n.

Ramsey numbers

Ramsey numbers are HARD to compute:

r \ s	2	3	4	5	6	7	8	9	
2	2	3	4	5	6	7	8	9	...
3		6	9	14	18	23	28	36	
4			18	25					

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"Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack." - Paul Erdős

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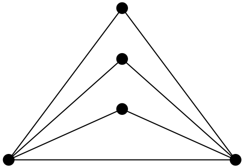
- $R(C_3, C_n) = 2n - 1$ for $n \geq 4$, $R(P_m, P_n) = n + \lfloor m/2 \rfloor + 1$
- Dynamic Survey (Radziszowski '94-present)

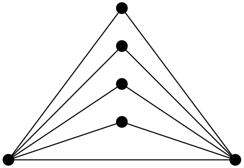
Book graphs

Definition

The book graph B_n is the graph $K_2 + \overline{K}_n$



B_3



B_4

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- Recent interest in asymptotics: Conlon '19, Conlon-Fox-Wigderson '23, Chen-Lin '24
- "Book algorithm" plays role in breakthrough proof that $R(s,s) \leq (4-\epsilon)^s$

Book graphs

Theorem (Rousseau-Sheehan '78)

For all n,

1. $R(B_n, B_n) = 4n + 2$ *if* $q = 4n + 1$ *is a prime power,*
2. $R(B_{n-1}, B_n) \leq 4n - 1$,
3. $R(B_{n-2}, B_n) \leq 4n - 3$ *if* $n \equiv 2 \pmod 3$.

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- Upper bounds come from Goodman's bound for number of triangles
- Lower bounds in (1) come from Paley graphs

New results on books

Theorem (W '24)

If $q = 2n - 1$ is a prime power congruent to 1 mod 4, then $R(B_{n-1}, B_n) = 4n - 1$.

Theorem (W'24, Lidický-McKinley-Pfender-Van Overberghe '24)

- $R(B_{n-2}, B_n) = 4n - 3$ *for* $n \leq 17$
- $R(B_{n-1}, B_n) = 4n - 1$ *for* $n \leq 20$
- $R(B_n, B_n) \geq 4n + 1$ *for* $n \leq 16$

New results on books and other graphs

Theorem (W '24, '25, Lidický-McKinley-Pfender-Van Overberghe '24)

Ramsey number	*old lower*	*old upper*	*new lower*	*new upper*
$R(B_2, B_8)$	*19*	*22*	**21**	**21**
$R(B_2, B_9)$	*21*	*24*	**22**	**22**
$R(B_2, B_{10})$	*23*	*26*	**25**	**25**
$R(B_2, B_{12})$	*27*	*29*	**28**	**28**
$R(B_2, B_{13})$	*29*	*31*	**29**	**29**
$R(B_3, B_7)$			**20**	**20**
$R(C_3, C_6, C_6)$	*15*	*18*		**15**
$R(C_5, C_6, C_6)$	*15*	*17*		**15**
$R(K_4, K_4 - e, K_4 - e)$	*33*	*47*	**35**	

Methods

- Boolean satisfiability (SAT) solvers have been used with great success in graph and arithmetic Ramsey theory!
  - $R(3,3,4) = 30$ (Codish-Frank-Itzhakov-Miller '16)
  - $R(G,G)$ for many sparse G (Low-Kapbasov-Kapbasov-Bereg '22)
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- Lower bound constructions from **circulant** graphs and generalizations

SAT encoding

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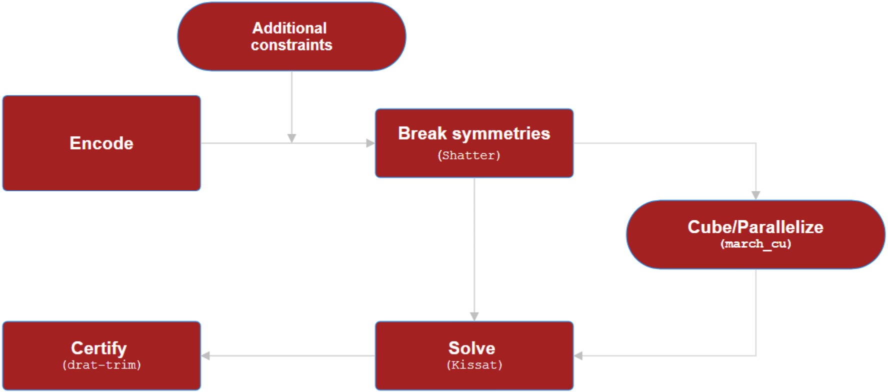
Satisfying assignments give **lower bounds**

Proofs of unsatisfiability give **upper bounds**

SAT toolchain



Circulant graphs

Definition

A graph $G = (V, E)$ on n vertices is circulant if there exists a set $S \subseteq \mathbb{Z}_n$ such that $\{x, y\} \in E$ if and only if $x - y \pmod{n} \in S$.

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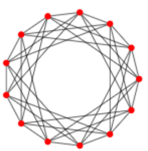
Example

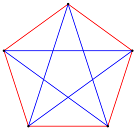
Paley graphs are circulant graphs with vertex set the finite field $\mathbb{F}_q$ for $q \equiv 1 \pmod{4}$ a prime power where S is the set of squares in $\mathbb{F}_q$.



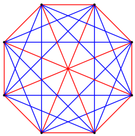
$q = 13,\ S = \{1, 4, 9, 3, 12, 10\}$

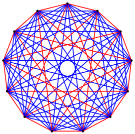
Lower bounds for small Ramsey numbers

$n = 5, \; S = \{1, 4\}$



$n = 8, \; S = \{1, 4, 7\}$





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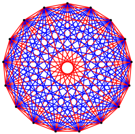

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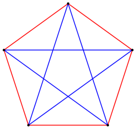

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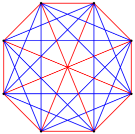

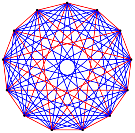

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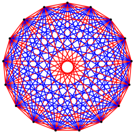

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Tight lower bounds for $R(3,3), R(3,4), R(3,5), R(4,4)$

Block-circulant graphs

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- Circulant graphs have circulant adjacency matrices
- A generalization:

Definition

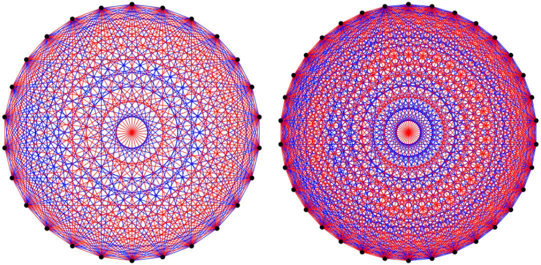
A graph is block-circulant (or *polycirculant* or *multicirculant*) if its adjacency matrix is of the form

$$A = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1k} \\ C_{21} & C_{22} & \dots & C_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ C_{k1} & C_{k2} & \dots & C_{kk} \end{pmatrix},$$

where each C_{ij} is a circulant matrix.

Block-circulant graphs

- Block-circulant graphs often yield good lower bounds for $R(K_r, K_s - e)$ (Goedgebeur-Van Overberghe '22)
- Many new lower bounds for diagonal or "nearly diagonal" book Ramsey numbers use 2-block constructions (Lidický-McKinley-Pfender-Van Overberghe '24, W'24)
- "2-block" version of Paley graphs gives $R(B_{n-1}, B_n) = 4n - 1$ (W '24)



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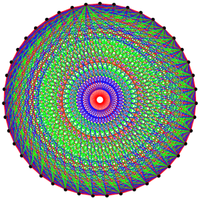

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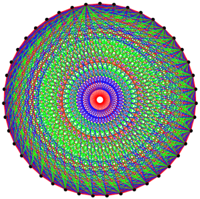

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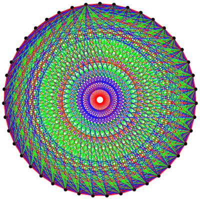

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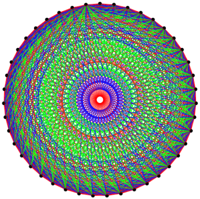

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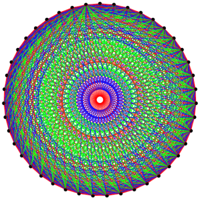

- Additional local search methods
- Dynamic symmetry breaking, CAS + SAT
- Finite geometry, other coloring problems
- Proof verification and minimization

Thank you!   Merci!