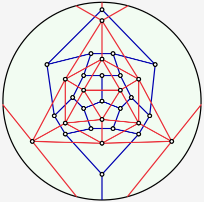
Verified encodings for SAT solvers

Cayden R. Codel
Advised by Marijn J. H. Heule and Jeremy Avigad



June 5, 2023

Repo at https://github.com/ccodel/verified-encodings

The SAT problem

SAT is an NP-hard problem in propositional logic

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$\tau = \{x_1,\ \overline{x}_2,\ x_3\}$

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$$F = (\mathbf{x_1} \lor x_2) \land (\overline{x}_1 \lor \mathbf{x_3}) \land (\overline{\mathbf{x}}_\mathbf{2} \lor \overline{x}_3)$$

$\tau = \{x_1,\ \overline{x}_2,\ x_3\}$

SAT solvers find a satisfying τ, or declare that none exists

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SAT solvers accept text input in conjunctive normal form

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```
p cnf 3 3
1 2 0
-1 3 0
-2 -3 0
```

SAT solvers at work

Hardware/software verification, optimization, SMT solvers

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Resolve longstanding problems in mathematics:

SAT solvers at work

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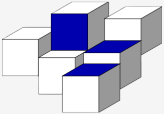
Keller's Conjecture



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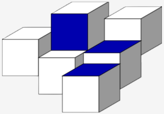

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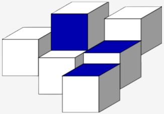


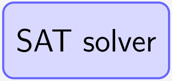
Pythagorean triples problem

$$a^2 + b^2 = c^2$$

Lam's Problem

1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0
0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0
0	1	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1

SAT toolchain



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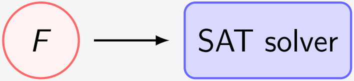

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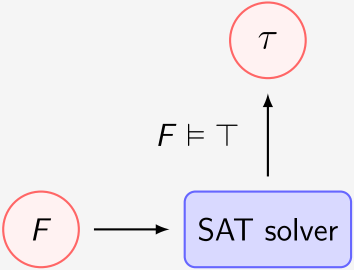

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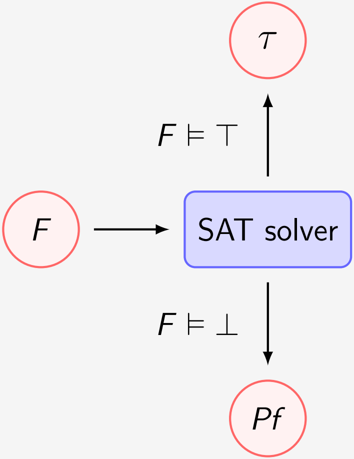

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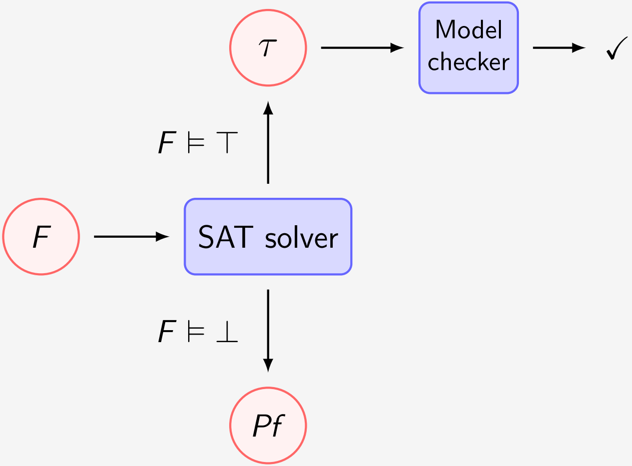

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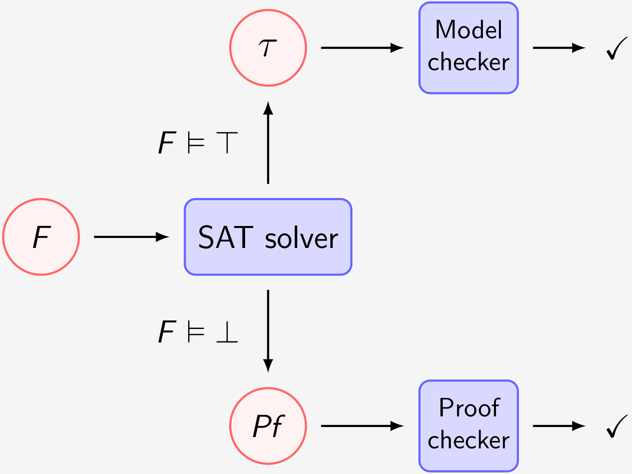

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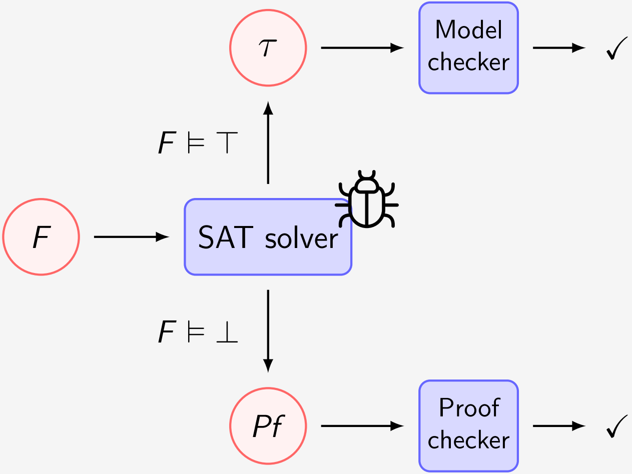

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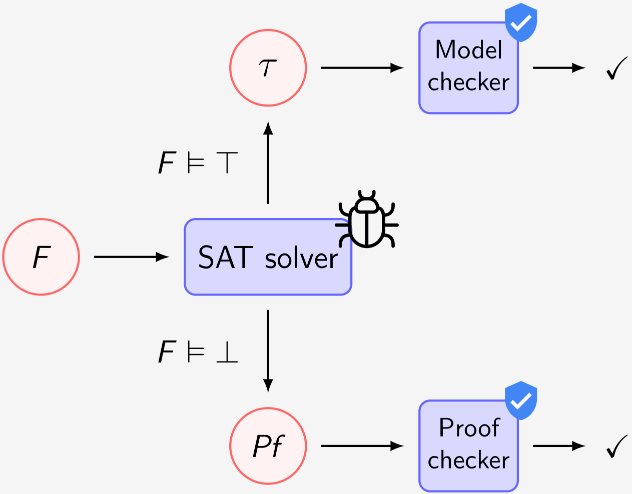

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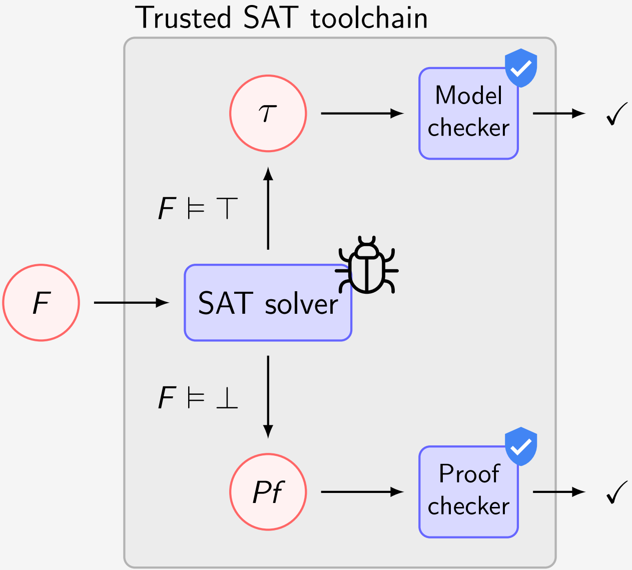

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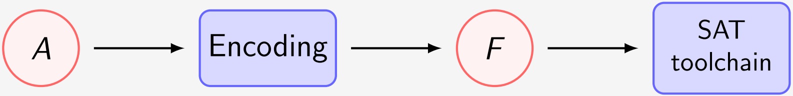

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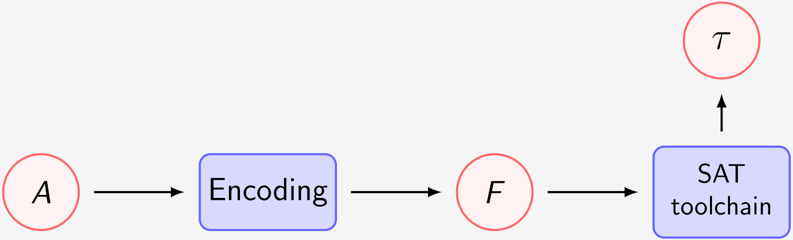

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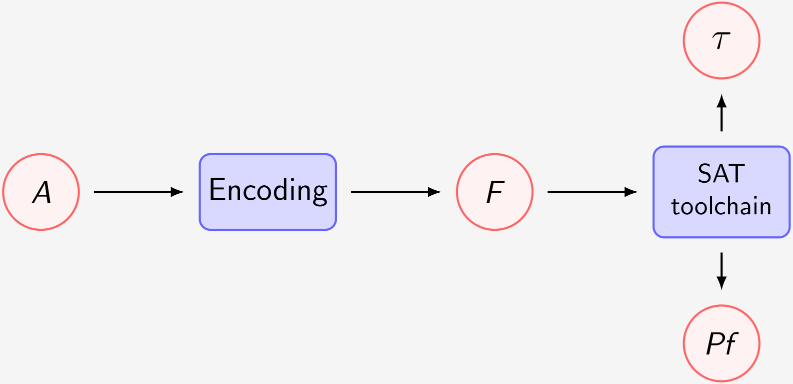

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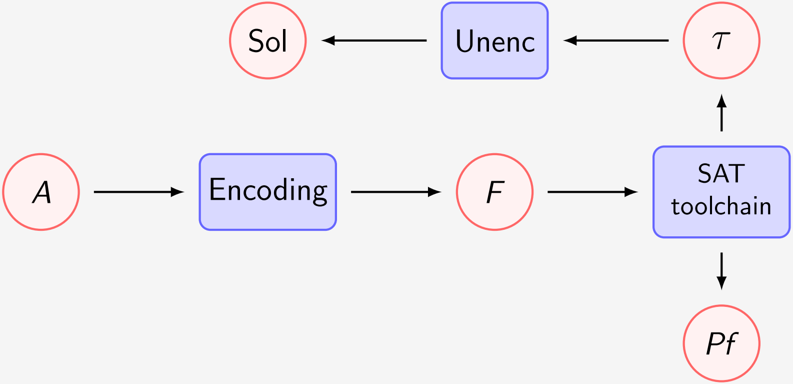

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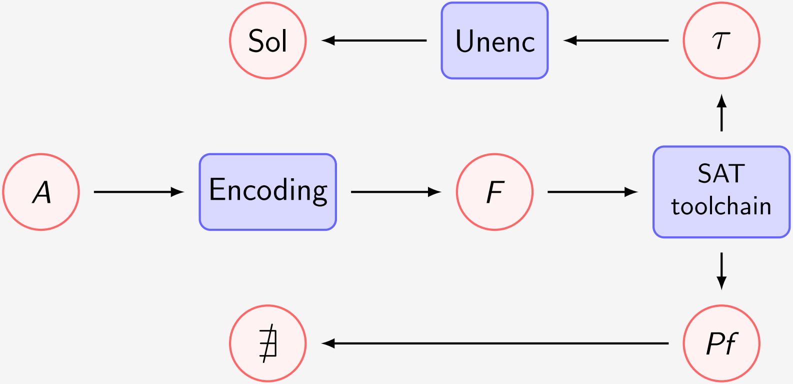

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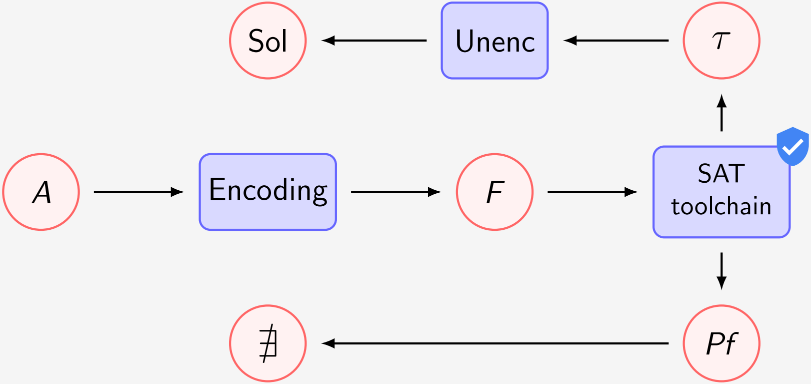

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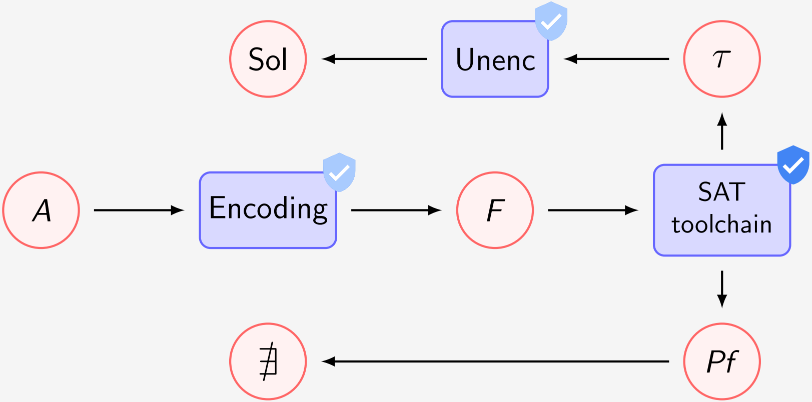

My work: extend the trusted SAT toolchain to include encodings by using a theorem prover

The Lean theorem prover

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We used version 3; version 4 is under active development

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theorem take_sublist_of_le {α : Type*} {i j : nat} : i ≤ j →
  ∀ (l : list α), l.take i <+ l.take j :=
begin
  intros hij l,
  induction l with a as ih generalizing i j,
  { rw [take_nil, take_nil] },
  { cases i,
    { rw take_zero,
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      { exact absurd hij (not_le.mpr (succ_pos i)) },
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Contains:

- Data structures (CNF representations, variable generation)
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- Proofs of correctness for parity, at-most-one, at-most-k
- Support for combining encodings to form larger ones

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Basis for future verification efforts

Library preliminaries

Goal: prove that an encoding is correct

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Q: What does it mean for an encoding to be correct?

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F encodes C if for all truth assignments τ,

$$C(\tau(x_1), \ldots, \tau(x_n)) \leftrightarrow \exists \sigma,\ \sigma(F) = \top,$$

where σ agrees with τ on X (i.e. $\forall x \in X,\ \tau(x) = \sigma(x)$)

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An encoding function E is correct for C if the formula it produces encodes C on all inputs

Library preliminaries

In Lean, the definitions look like:

```
def encodes (C : constraint) (l : list literal) (F : cnf) :=
  ∀ (τ : assignment),
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```
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We prove that the encodings in our library are correct and well-behaved (generate new variables in a reasonable manner)

Case study: at-most-one

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The Sinz encoding produces $O(n)$ clauses and needs $n - 1$ new variables:

$$\mathrm{Sinz}(X) = \bigwedge_{i=1}^{n-1} \left((\overline{x}_i \vee s_i) \wedge (\overline{s}_i \vee s_{i+1}) \wedge (\overline{s}_i \vee \overline{x}_{i+1}) \right)$$

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The three clauses are logically equivalent to

$$(x_i \to s_i) \wedge (s_i \to s_{i+1}) \wedge (s_i \to \overline{x}_{i+1})$$

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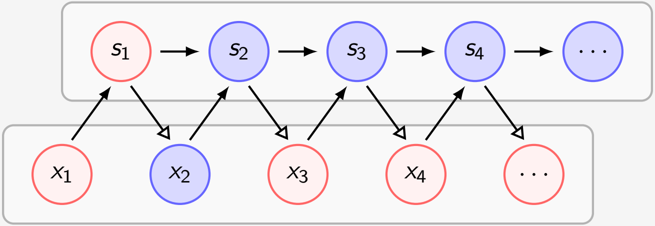

(Hollow arrow heads indicate negated implications)

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Encodings in Lean's functional programming language:

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| [l₁, l₂]          g :=
  let ⟨y, g₁⟩ := g.fresh in
  ⟨[[l₁.flip, Pos y], [Neg y, l₂.flip]], g₁⟩

| (l₁ :: l₂ :: ls) g :=
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      [Neg y, l₂.flip]] ++ F_rec, g₂⟩
```

Case study: at-most-one

Encodings in Lean's functional programming language:

```
def Sinz_amo : enc_fn
| [l₁, l₂]          g :=
  let ⟨y, g₁⟩ := g.fresh in
  ⟨[[l₁.flip, Pos y], [Neg y, l₂.flip]], g₁⟩

| (l₁ :: l₂ :: ls) g :=
  let ⟨y, g₁⟩ := g.fresh in
  let ⟨z, _⟩ := g₁.fresh in
  let ⟨F_rec, g₂⟩ := sinz_rec (l₂ :: ls) g₁ in
  ⟨[[l₁.flip, Pos y], [Neg y, Pos z],
      [Neg y, l₂.flip]] ++ F_rec, g₂⟩
```

Applications and future work

Combine sub-encodings to form more complex ones

Easily recover proofs of correctness

```
def append (enc₁ enc₂ : enc_fn) : enc_fn :=
  λ (l : list literal) (g : gensym),
  let (F₁, g₁) := enc₁ l g in
  let (F₂, g₂) := enc₂ l g₁ in
  (F₁ ++ F₂, g₂)
```

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```

```
theorem is_correct_append
  {c₁ c₂ : constraint} {enc₁ enc₂ : enc_fn V} :
  is_correct c₁ enc₁ → is_correct c₂ enc₂ →
  is_correct (c₁ ++ c₂) (enc₁ ++ enc₂) := ...
```

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  λ (l : list literal) (g : gensym),
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  let (F₂, g₂) := enc₂ l g₁ in
  (F₁ ++ F₂, g₂)
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```
theorem is_correct_append
  {c₁ c₂ : constraint} {enc₁ enc₂ : enc_fn V} :
  is_correct c₁ enc₁ → is_correct c₂ enc₂ →
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```

Already demonstrated by combining sub-encodings for Sudoku

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- Prove more (sub-)encodings correct
- Prove the Keller reduction correct
- Write verified proof checkers for SAT proof systems

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- Prove more (sub-)encodings correct
- Prove the Keller reduction correct
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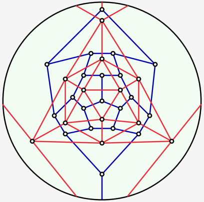
Overall, the goal is to make Lean the one-stop-shop for generating SAT queries in a trusted way



Thank you!
Any questions?