A Computational Counterexample on Sets Containing Fibonacci Numbers

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CanaDAM 2025
Computer-Assisted Mathematics
May 20, 2025, University of Ottawa

Part I: The Problem & Patterns

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  Prove or disprove that each generation contains at least one Fibonacci number or its negative.

- We call G_i generation *i* and we call *i* the generation index.
- Note that the sets G_i are pairwise disjoint as they consist only of new elements added to $\mathcal{S}$.

- We can in fact start with $G_0 = \{0\}$ and apply the rules of the problem.

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\begin{aligned}
G_0 &= \{0\} \\
G_1 &= \{1\} \\
G_2 &= \{2\} \\
G_3 &= \{\text{-}1, 4\} \\
G_4 &= \{\text{-}3, \text{-}2, 8\} \\
G_5 &= \{\text{-}7, \text{-}6, \text{-}4, 3, 16\} \\
G_6 &= \{\text{-}15, \text{-}14, \text{-}12, \text{-}8, 5, 6, 7, 32\} \\
G_7 &= \{\text{-}31, \text{-}30, \text{-}28, \text{-}24, \text{-}16, \text{-}5, 9, 10, 12, 13, 14, 15, 64\}
\end{aligned}
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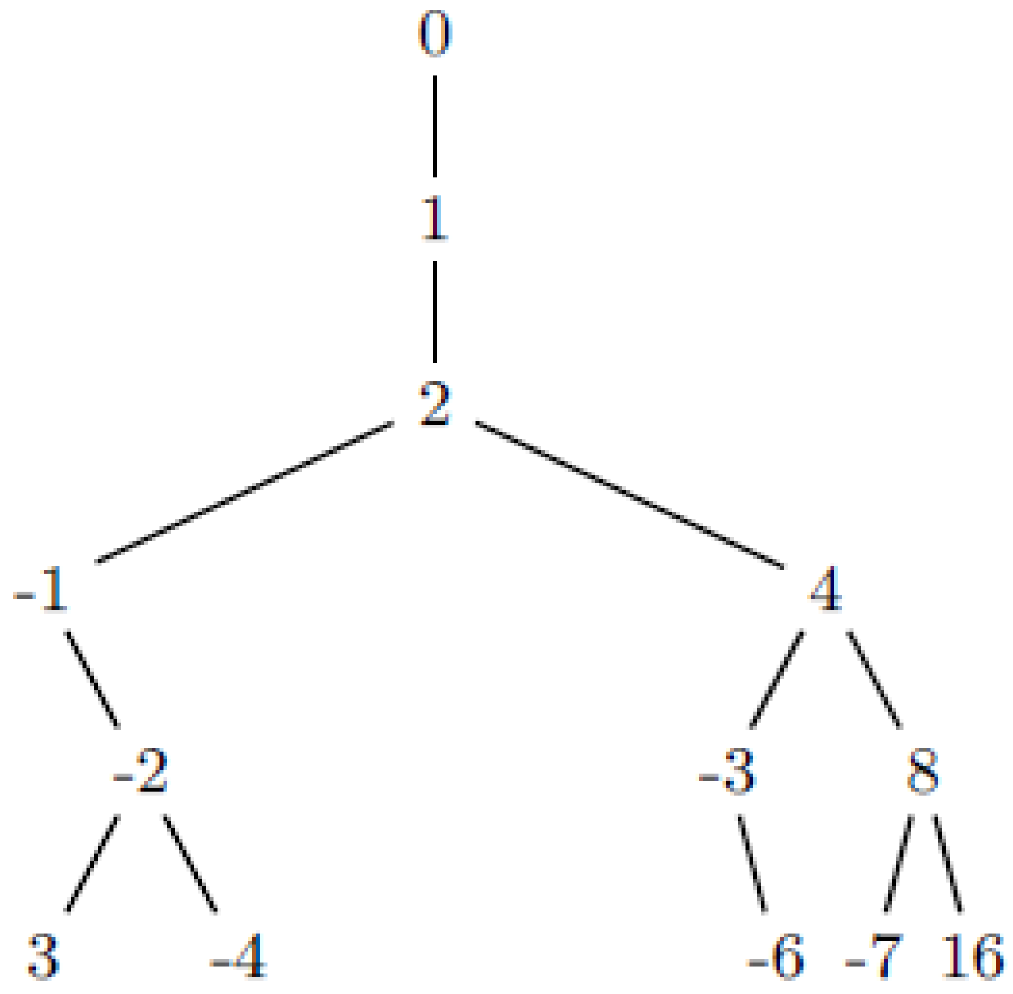
G_2 &= \{2\} \\
G_3 &= \{-1, 4\} \\
G_4 &= \{-3, -2, 8\} \\
G_5 &= \{-7, -6, -4, 3, 16\} \\
G_6 &= \{-15, -14, -12, -8, 5, 6, 7, 32\} \\
G_7 &= \{-31, -30, -28, -24, -16, -5, 9, 10, 12, 13, 14, 15, 64\}
\end{aligned}
$$

- So far each generation contains at least one Fibonacci number or its negative!

- We can easily visualize the growth of $\mathcal{S}$ using a binary tree:



- Structure in the tree:

  Theorem 1: For $i \geq 3$ the even elements of G_i (right children) come from doubling all terms of G_{i-1} , and the odd elements of G_i (left children) come from subtracting the double of all terms of G_{i-2} from 1.

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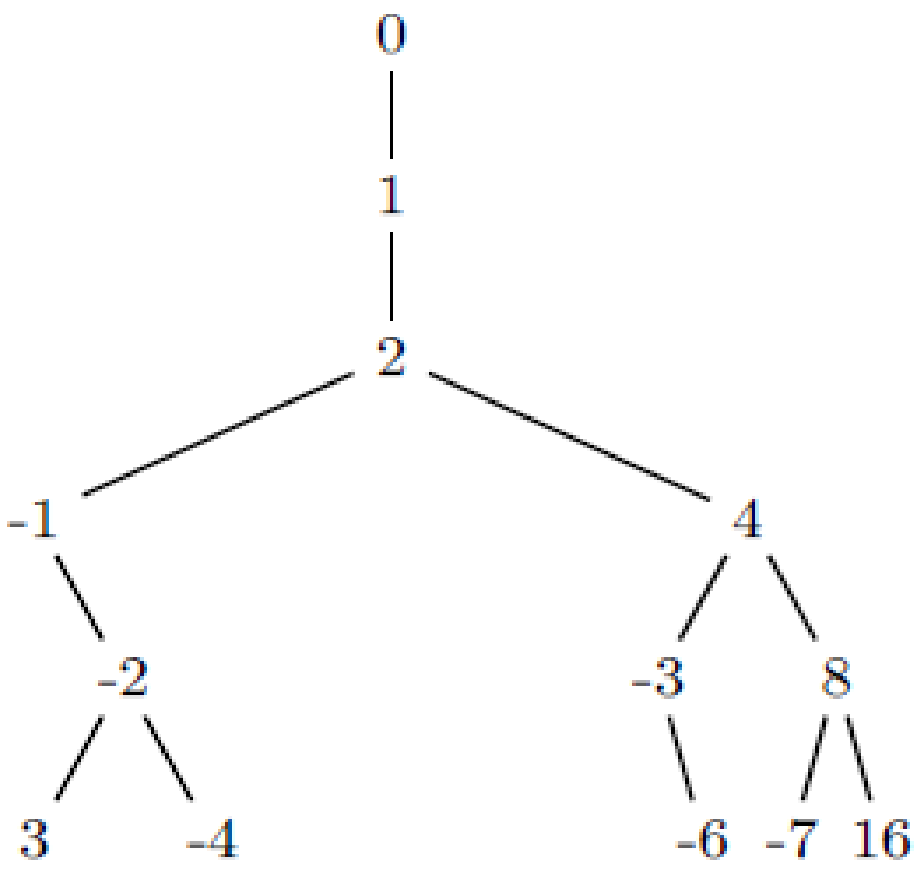

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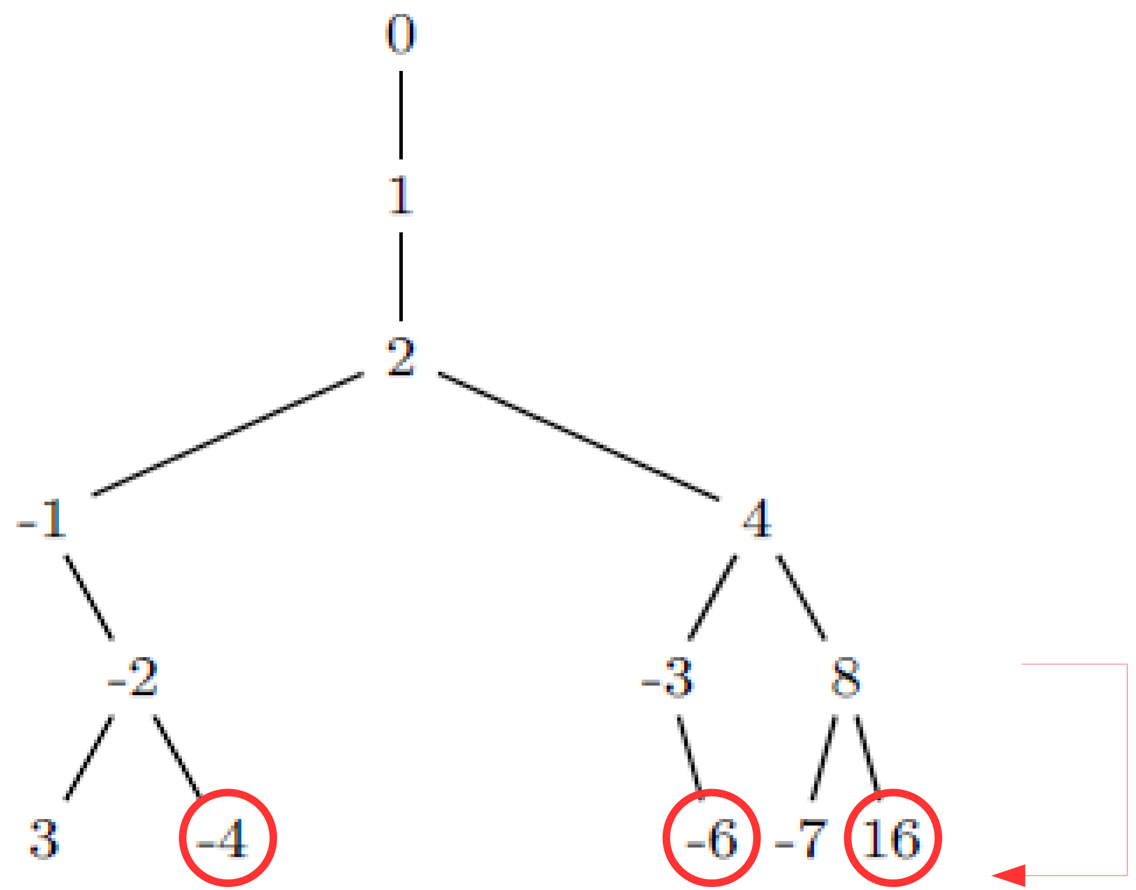

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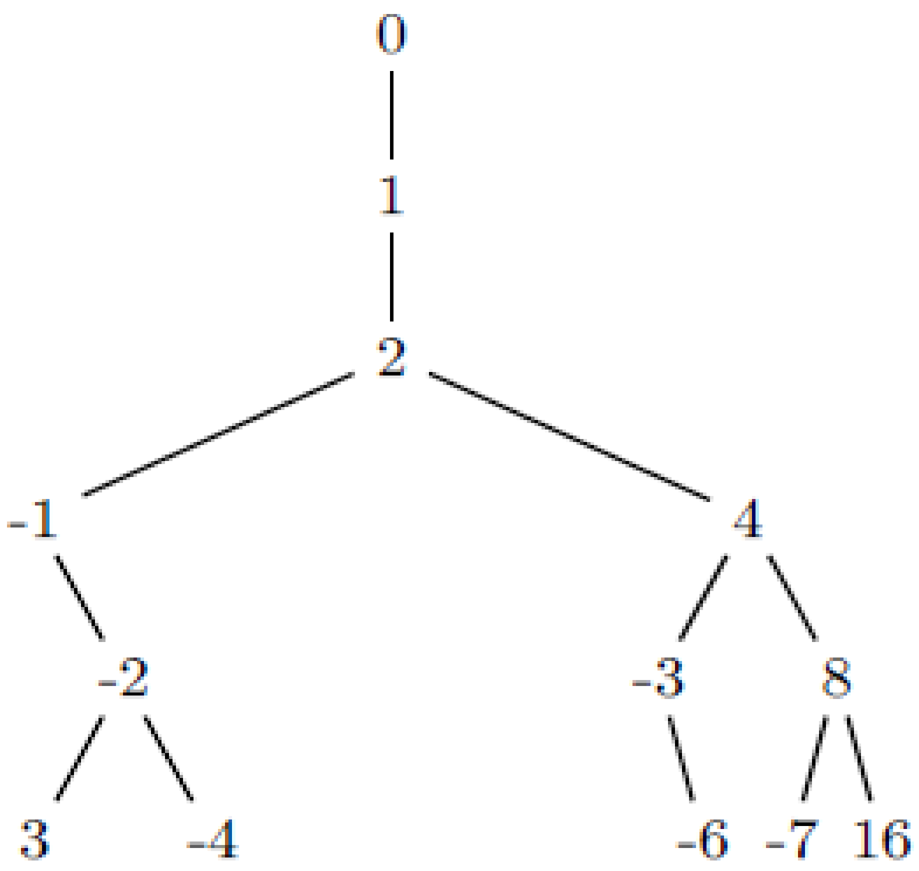

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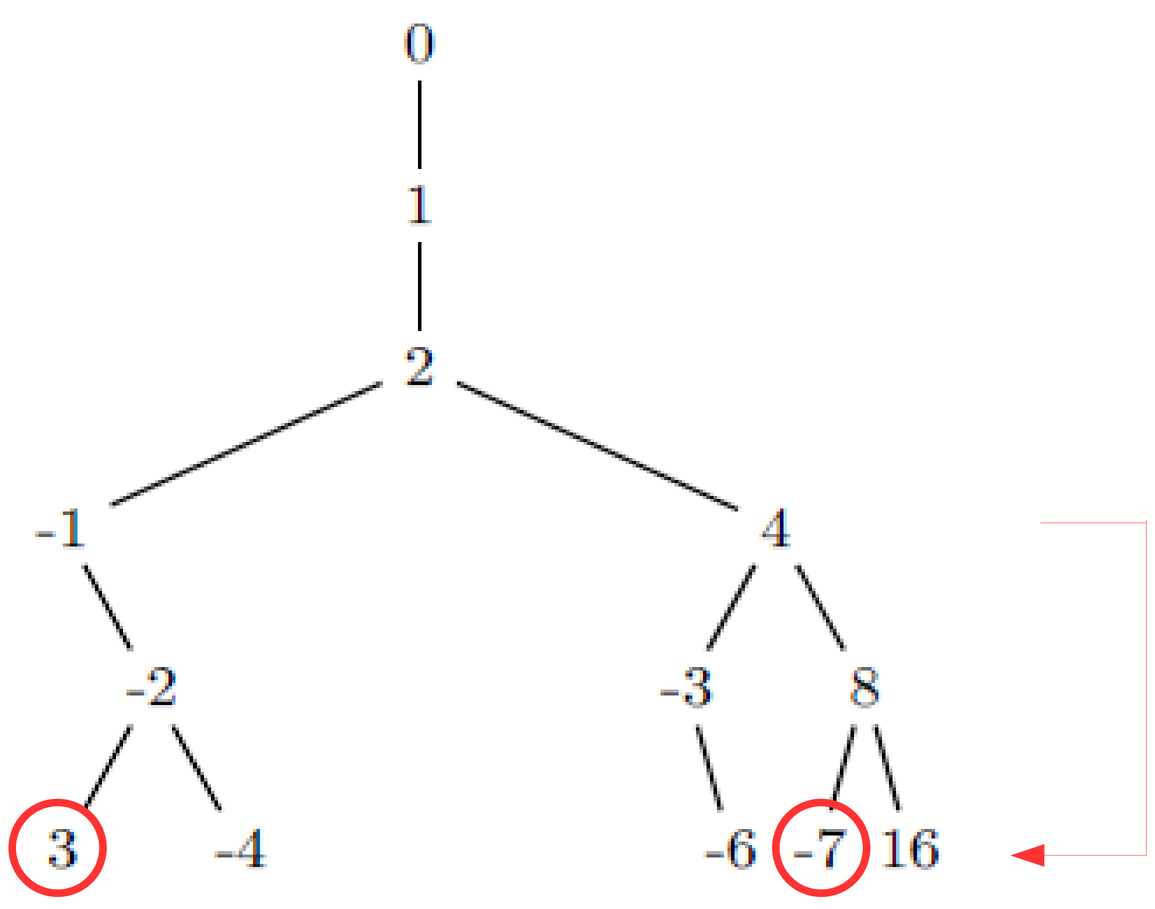

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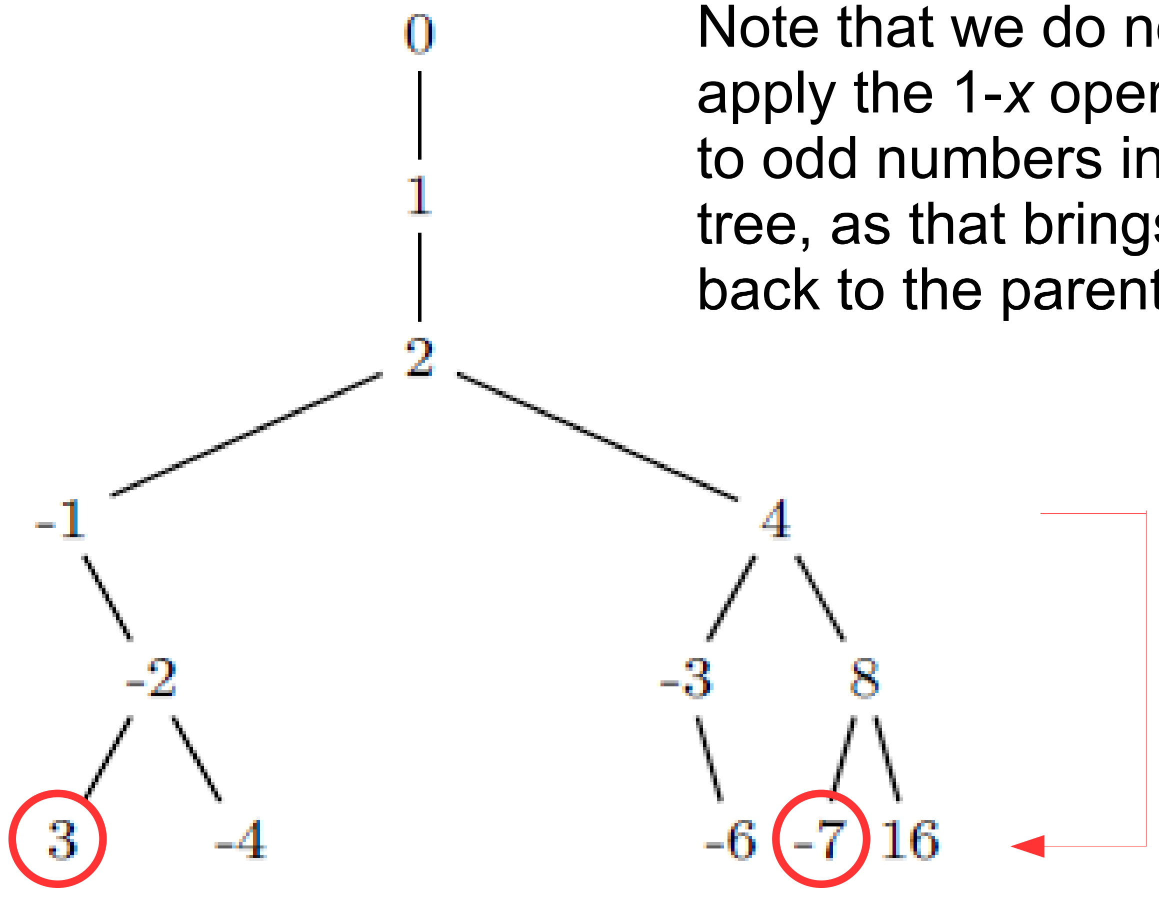

Note that we do not apply the 1-*x* operation to odd numbers in the tree, as that brings us back to the parent!

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- The number of terms in each row is growing exponentially, at rate approximately the golden ratio, ϕ = 1.618033...
- After a failed cursory search for a counterexample, we moved to other methods.

- The first questions we asked were:
  - What numbers *do* appear in $\mathcal{S}$?
  - Where do they appear in the tree?

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- Theorem 2: All integers belong to the set $\mathcal{S}$.

  Idea of proof: We can use a contradiction, assuming some integer *k* is not in $\mathcal{S}$. We can then trace a decreasing sequence (in absolute value) of uniquely appearing integers from *k* back to the node of our tree using the two operations defined in Theorem 1

- The following table gives the generation index *i* for integers $|k| \leq 15$:

k	i	k	i	k	i	k	i
		2	**2**	4	**3**	6	**6**
0	**0**	-2	**4**	-4	**5**	-6	**5**
1	**1**	3	**5**	5	**6**	7	**6**
-1	**3**	-3	**4**	-5	**7**	-7	**5**

k	i	k	i	k	i	k	i
8	**4**	10	**7**	12	**7**	14	**7**
-8	**6**	-10	**8**	-12	**6**	-14	**6**
9	**7**	11	**9**	13	**7**	15	**7**
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-1	**3**	-3	**4**	-5	**7**	-7	**5**

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9	**7**	11	**9**	13	**7**	15	**7**
-9	**8**	-11	**8**	-13	**8**	-15	**6**

- The sequence of indices was not in OEIS!

- We will now uncover patterns found in the table.

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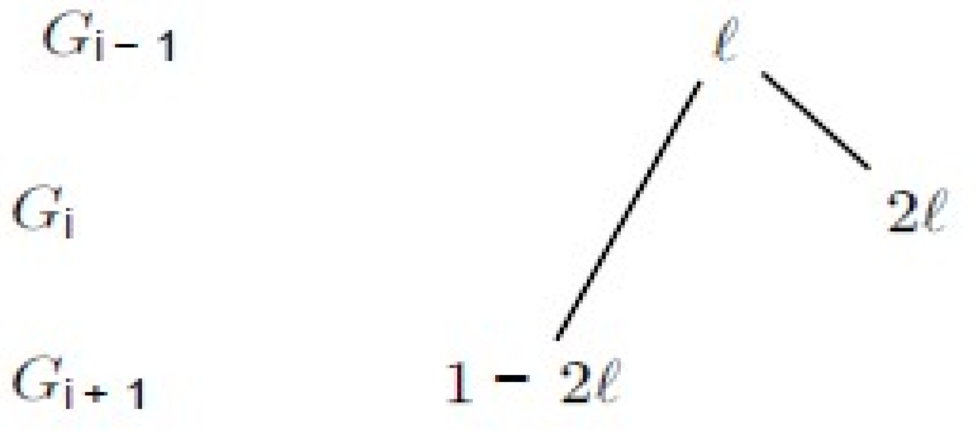
- Theorem 3: When moving from a negative odd number ($1-2\ell$) to a positive even number (2ℓ) in the table, the generation index decreases by 1.
- Idea of Proof: Given a positive integer ℓ, its double 2ℓ appears in the next generation and $1-2\ell$ appears two generations later, by the operations defined in the problem.



k	i	k	i	k	i	k	i
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1	**1**	3	**5**	5	**6**	7	**6**
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-9	**8**	-11	**8**	-13	**8**	-15	**6**

- Theorem 4: When moving from a negative even number to a positive odd number in the table, the generation index increases by 1.

k	i	k	i	k	i	k	i
		2	**2**	4	**3**	6	**6**
0	**0**	-2	**4**	-4	**5**	-6	**5**
1	**1**	3	**5**	5	**6**	7	**6**
-1	**3**	-3	**4**	-5	**7**	-7	**5**

k	i	k	i	k	i	k	i
8	**4**	10	**7**	12	**7**	14	**7**
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9	**7**	11	**9**	13	**7**	15	**7**
-9	**8**	-11	**8**	-13	**8**	-15	**6**

- Theorem 5: When moving from a positive odd number *k* to its negative in the table, the generation index:
  increases by 1 if $k = 1 \pmod 4$
  decreases by 1 if $k = 3 \pmod 4$

k	i	k	i	k	i	k	i
		2	**2**	4	**3**	6	**6**
0	**0**	-2	**4**	-4	**5**	-6	**5**
1	**1**	3	**5**	5	**6**	7	**6**
-1	**3**	-3	**4**	-5	**7**	-7	**5**

k	i	k	i	k	i	k	i
8	**4**	10	**7**	12	**7**	14	**7**
-8	**6**	-10	**8**	-12	**6**	-14	**6**
9	**7**	11	**9**	13	**7**	15	**7**
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- Theorem 6: When moving from a positive even number $2^i m$ (where $m \geq 3$, odd) to its negative in the table, the generation index:

  increases by 1 if $m = 1 \pmod 4$

  decreases by 1 if $m = 3 \pmod 4$

k	i	k	i	k	i	k	i
		2	**2**	4	**3**	6	**6**
0	**0**	-2	**4**	-4	**5**	-6	**5**
1	**1**	3	**5**	5	**6**	7	**6**
-1	**3**	-3	**4**	-5	**7**	-7	**5**

m=3

k	i	k	i	k	i	k	i
8	**4**	10	**7**	12	**7**	14	**7**
-8	**6**	-10	**8**	-12	**6**	-14	**6**
9	**7**	11	**9**	13	**7**	15	**7**
-9	**8**	-11	**8**	-13	**8**	-15	**6**

m=5 **m=3** **m=7**

- Theorem 7: When moving from a postive power of 2 to its negative in the table, the generation index increases by 2.

k	i	k	i	k	i	k	i
		2	**2**	4	**3**	6	**6**
0	**0**	-2	**4**	-4	**5**	-6	**5**
1	**1**	3	**5**	5	**6**	7	**6**
-1	**3**	-3	**4**	-5	**7**	-7	**5**

k	i	k	i	k	i	k	i
8	**4**	10	**7**	12	**7**	14	**7**
-8	**6**	-10	**8**	-12	**6**	-14	**6**
9	**7**	11	**9**	13	**7**	15	**7**
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Part II: An Expression for the Generation Index

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- This means that

$$n = \begin{cases} 2k - 1, & k > 0; \\ -2k, & k \leq 0, \end{cases}$$

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k	n	f(n)	k	n	f(n)	k	n	f(n)	k	n	f(n)
			2	3	**2**	4	7	**3**	6	11	**6**
0	0	**0**	-2	4	**4**	-4	8	**5**	-6	12	**5**
1	1	**1**	3	5	**5**	5	9	**6**	7	13	**6**
-1	2	**3**	-3	6	**4**	-5	10	**7**	-7	14	**5**

- We will redefine the generation index *i* as the sequence $f(n)$ for *n* ≥ 0.
- This means that

$$n = \begin{cases} 2k-1, & k > 0; \\ -2k, & k \leq 0, \end{cases}$$

- So, integer *k* is found in generation $f(n)$.

k	n	f(n)	k	n	f(n)	k	n	f(n)	k	n	f(n)
			2	3	**2**	4	7	**3**	6	11	**6**
0	0	**0**	-2	4	**4**	-4	8	**5**	-6	12	**5**
1	1	**1**	3	5	**5**	5	9	**6**	7	13	**6**
-1	2	**3**	-3	6	**4**	-5	10	**7**	-7	14	**5**

- Example: *k* = -5 corresponds to *n* = 10 and is in generation $f(10) = 7$.

- Consider the difference sequence of $f(n)$, which we will denote $f_d(n)$ for $n \geq 1$ and define $f_d(0) = 0$.

$$f_d(n) = f(n) - f(n-1)$$

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- This sequence is given in the table below, read column-wise:

0	2	2	-1	2	1	-1	-1
1	1	1	1	1	1	1	1
2	-1	1	-1	1	-1	1	-1
-1	-1	-1	-1	-1	-1	-1	-1

- Consider the difference sequence of $f(n)$, which we will denote $f_d(n)$ for *n* ≥1 and define $f_d(0) = 0$.

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0	2	2	-1	2	1	-1	-1
1	1	1	1	1	1	1	1
2	-1	1	-1	1	-1	1	-1
-1	-1	-1	-1	-1	-1	-1	-1

- Neither sequence $f(n)$ nor $f_d(n)$ was found in OEIS. However.... similar sequences were!

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0	1	1	-1	1	1	-1	-1
1	1	1	1	1	1	1	1
1	-1	1	-1	1	-1	1	-1
-1	-1	-1	-1	-1	-1	-1	-1

- Notice that $f_d(n) = 2$ for $n = 2^k, k \geq 1$.
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1	1	1	1	1	1	1	1
1	-1	1	-1	1	-1	1	-1
-1	-1	-1	-1	-1	-1	-1	-1

- This is OEIS sequence A034947 for *n* ≥1!

- Sequence A034947 is the Jacobi symbol (-1/*n*), and is given by the recursion:

$$
\begin{aligned}
\alpha(4n+3) &= -1, & n &\geq 0; \\
\alpha(4n+1) &= 1, & n &\geq 0; \\
\alpha(2n) &= \alpha(n), & n &\geq 1.
\end{aligned}
$$

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- We can verify that $a_d(n)$ matches A034947 by using Theorems 3 – 7 to prove this recursion.

  (The sequence $f_d(n)$ also follows this recursion.)

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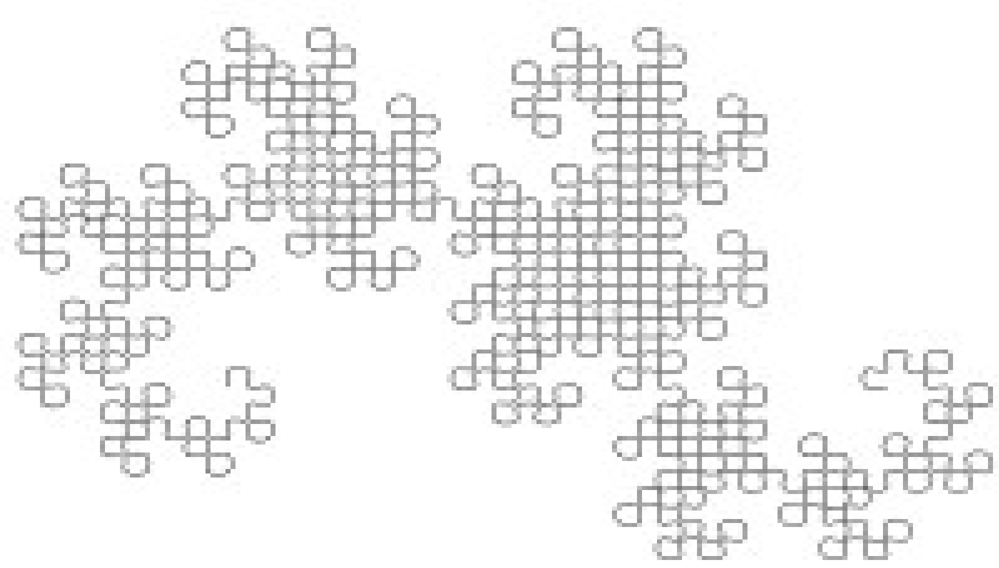

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$$a_d(n) = a(n) - a(n-1)$$

$a_d(n)$: 0, 1, 1, -1, 1, 1, -1, -1, 1, 1, 1, -1, -1, 1, ...

$a(n)$: 0, 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 3, ...

- So now what?
- We started with $f(n)$ and looked at the difference sequence $f_d(n)$.
- We then divided out the powers of 2 to get $a_d(n)$.
- Let's now take the partial sums of the sequence $a_d(n)$, and denote this sequence by $a(n)$ for *n* ≥ 1, and let $a(0) = 0$:

$$a_d(n) = a(n) - a(n-1)$$

$a_d(n)$: 0, 1, 1, -1, 1, 1, -1, -1, 1, 1, 1, -1, -1, 1, ...

$a(n)$: 0, 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 3, ...

- $a(n)$ is OEIS sequence A005811!

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- Sequence A005811 is the number of runs in the binary expansion of *n* for *n* ≥ 1. (Also, it is the number of 1s in the Gray code of *n*.)
- This is easily calculable (and non-recursive!)

k	**n**	**a(n)**	**binary**
0	0	0	
1	1	1	1
-1	2	2	10
2	3	1	11
-2	4	2	100
3	5	3	101
-3	6	2	110
4	7	1	111
-4	8	2	1000
5	9	3	1001
-5	10	4	1010

- The sequence $a(n)$ almost gives us $f(n)$.
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- Recall that this gives us the generation index for any integer *k*, where *n* ≥ 1, $f(0) = 0$ and

$$n = \begin{cases} 2k - 1, & k > 0; \\ -2k, & k \leq 0, \end{cases}$$

- We can now easily calculate the generation at which any integer will appear in the binary tree!

k	n	a(n)	$\lfloor log_2(n) \rfloor$	f(n)
0	0	0		0
1	1	1	0	1
-1	2	2	1	3
2	3	1	1	2
-2	4	2	2	4
3	5	3	2	5
-3	6	2	2	4
4	7	1	2	3
-4	8	2	3	5
5	9	3	3	6
-5	10	4	3	7

Part III: The Counterexample and Further Work

- Recall: we are interested in which generations contain a Fibonacci number or its negative.
- We will restrict our values of k to these numbers only!

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- We will restrict our values of *k* to these numbers only!
- The following table gives the generation indices for the first 64 Fibonacci numbers and their negatives:

k = 1, -1, 1, -1, 2, -2, 3, -3, 5, -5, 8, -8, ...

1	3	1	3	2	4	5	4	6	7	4	6
7	8	10	11	9	10	11	10	12	13	11	12
13	14	14	15	15	16	17	16	16	17	19	18
22	23	22	23	23	24	26	25	20	21	25	26
28	29	26	27	25	26	32	31	30	31	31	32
36	37	33	34	31	32	36	35	37	38	35	34
36	37	41	42	39	40	42	41	45	46	35	34
48	49	49	50	44	45	50	49	49	50	48	49
48	49	51	52	54	55	52	51	55	56	58	59
57	58	63	64	64	65	65	64	59	60	62	63
69	70	63	64	62	63	68	69				

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- So it seems these generations may not contain a Fibonacci number or its negative!
  - But can we be sure these numbers won't arise later in the table?

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- Recall that: $f(n) = a(n) + \lfloor log_2(n) \rfloor$
- $a(n)$ is positive and neither increasing nor decreasing.
- $\lfloor log_2(n) \rfloor$ is non-decreasing, so it provides a lower bound: $f(n) \geq \lfloor log_2(n) \rfloor$

- Let $k = F_{64} = 10,610,209,857,723$

  Therefore $n = 2k - 1 = 21,220,419,715,445$

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- This means that any Fibonacci number past the 64th must occur in generation 44 or higher.
- Therefore 43 will never appear in the table, and generation 43 does not contain a Fibonacci number or its negative!

- Note: generation 43 contains $F_{43} = 433,494,437$ integers, so finding this counterexample using brute force would have been computationally difficult.

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- Further, $f(n)$ can be expressed by the following recurrence, which would also have been cumbersome to use, as we are only interested in the Fibonacci cases:

$$f(2n) = f(n) + \begin{cases} 1, & n \ \textit{even}; \\ 2, & n \ \textit{odd}. \end{cases}$$

$$f(2n+1) = f(n) + \begin{cases} 1, & n \ \textit{odd}; \\ 2, & n \ \textit{even}. \end{cases}$$

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- The first several such generations are: 43, 47, 53, 61, 66, 67, 73, 82, 107, 108, 124, 143, 150, ...
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  - Of these, 509 generations, or 14.66% do not contain a Fibonacci number or its negative!
- 14.66% of the first 4161 generations do not.
- 14.70% of the first 4865 generations do not.

- We also considered generalized Fibonacci sequences -- starting with terms $1, a$, where $a \in \mathbb{Z}, 1 \leq a \leq 12$.
- We found that between 13% and 15% of generations failed to contain a number in the sequence or its negative.

Thank you!!