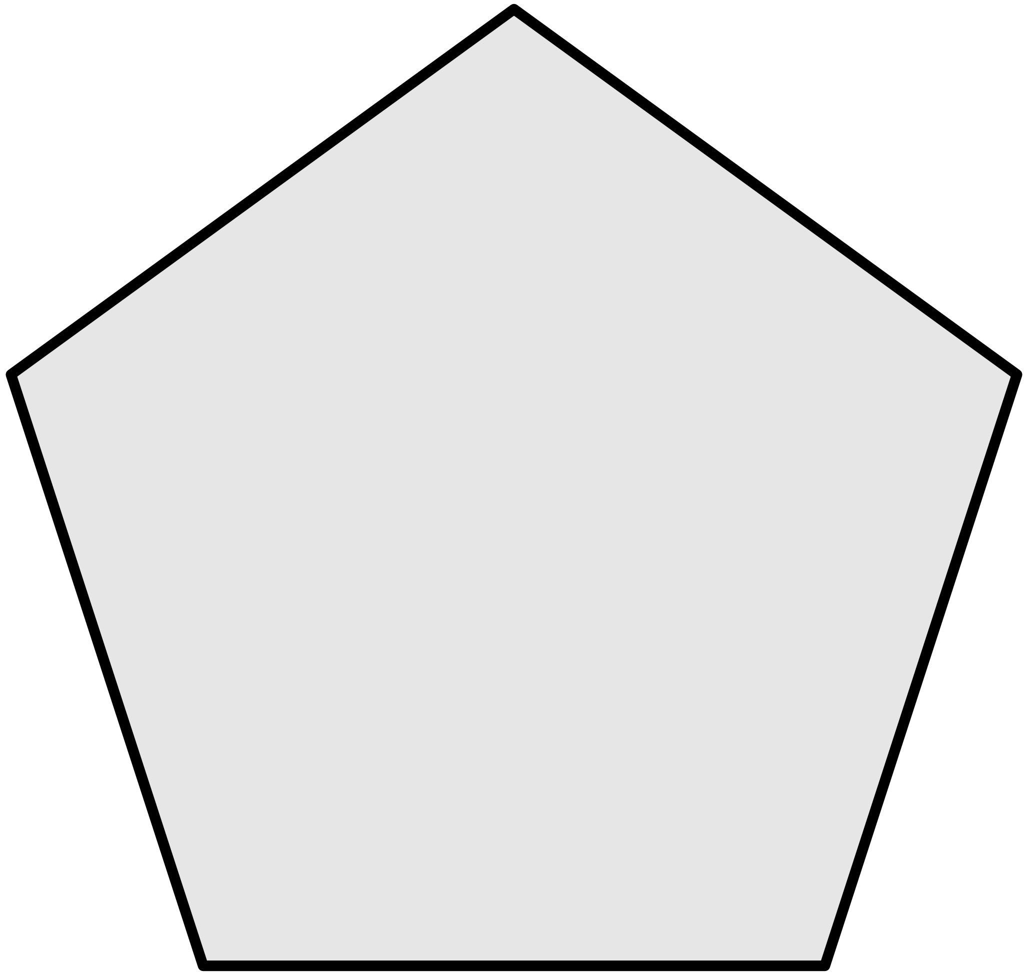
Heesch Numbers of Unmarked Polyforms

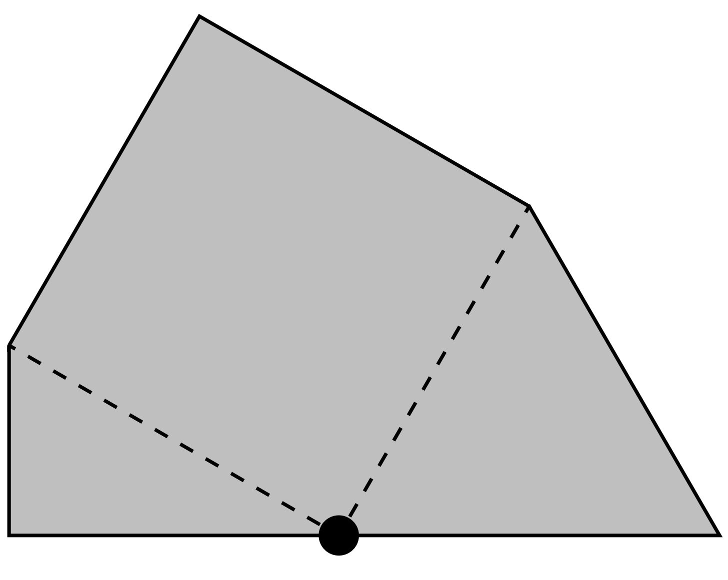
Craig S. Kaplan, University of Waterloo

Canadam 2023 ▪ 5 June 2023

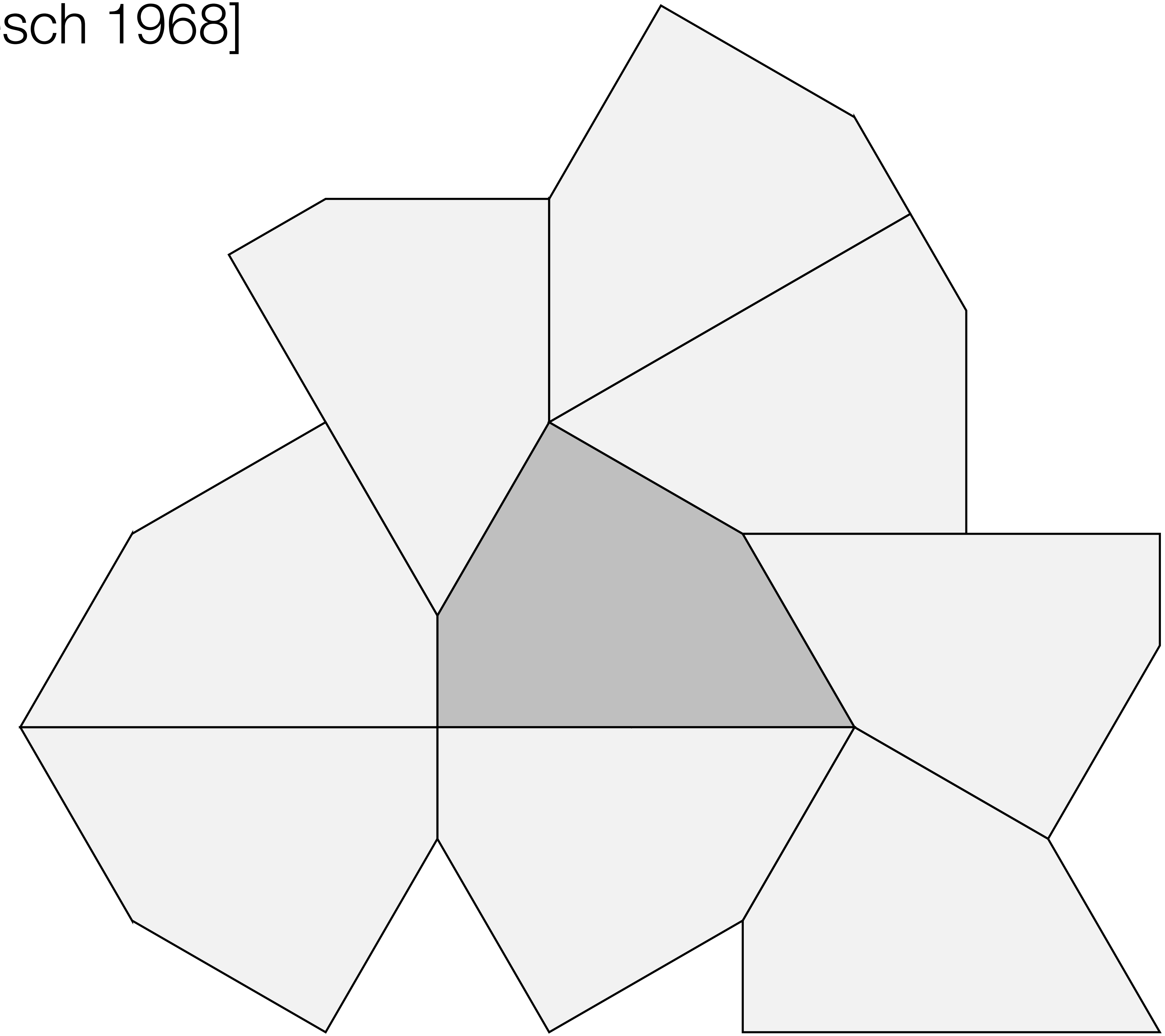
(Non-)Tiling Theory

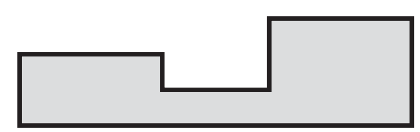
If a shape doesn't tile the plane, how convincingly can it pretend that it does?

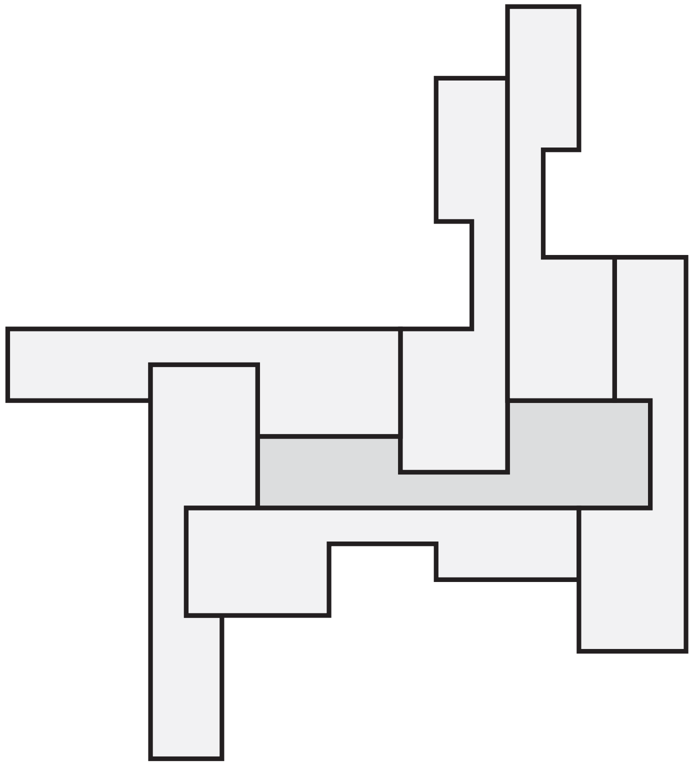




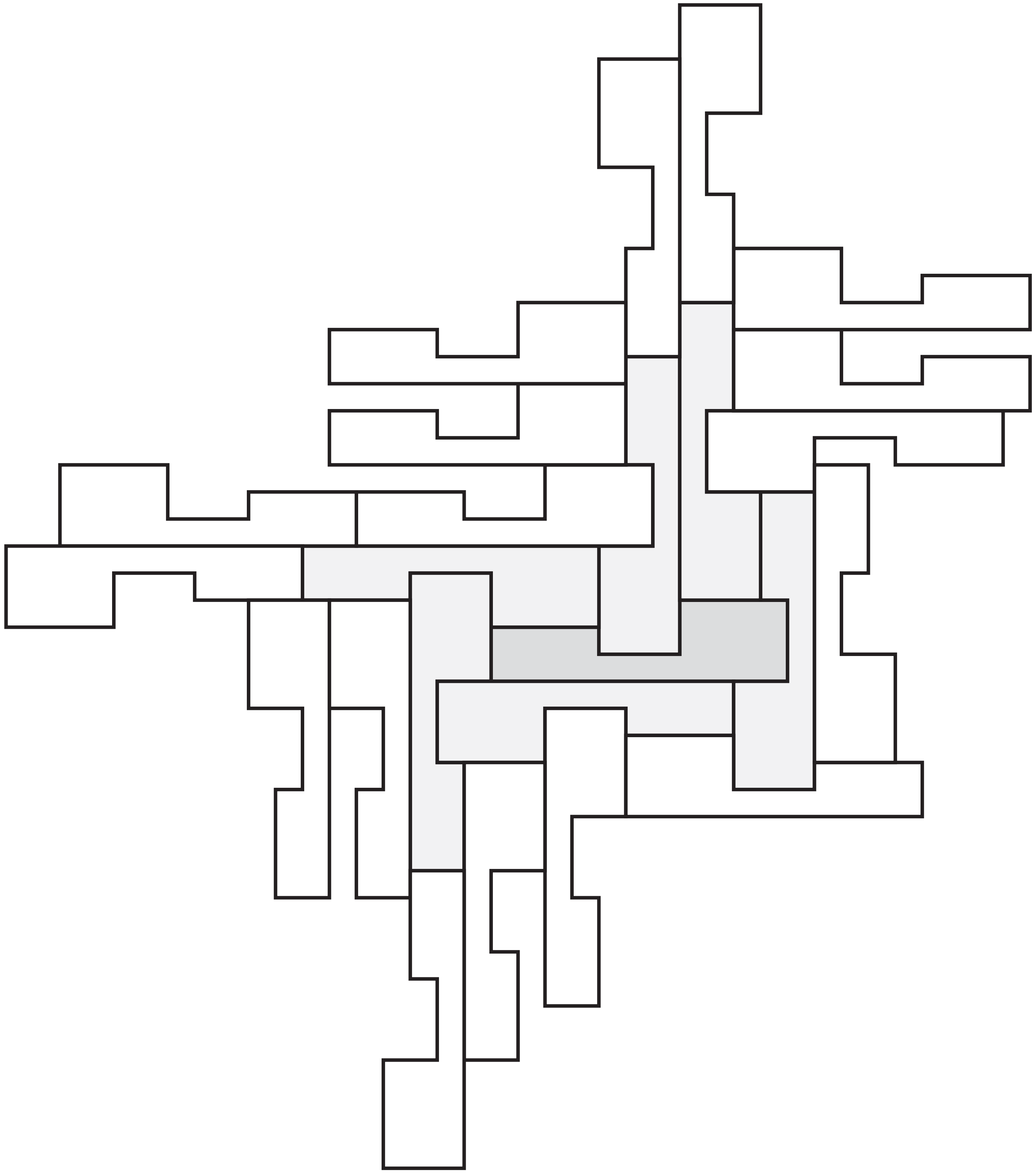
[Heesch 1968]







[Fontaine 1991]



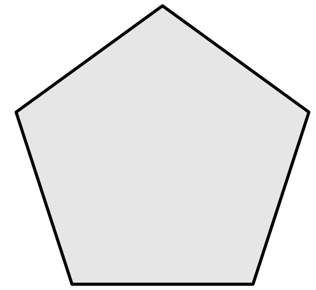
A **surround** of C by S is a set $\{S_1, \ldots, S_n\}$ of copies of S such that

- $P = C \cup S_1 \cup \ldots \cup S_n$ is a topological disk;
- C is in the interior of P; and
- C and the S_i have pairwise disjoint interiors

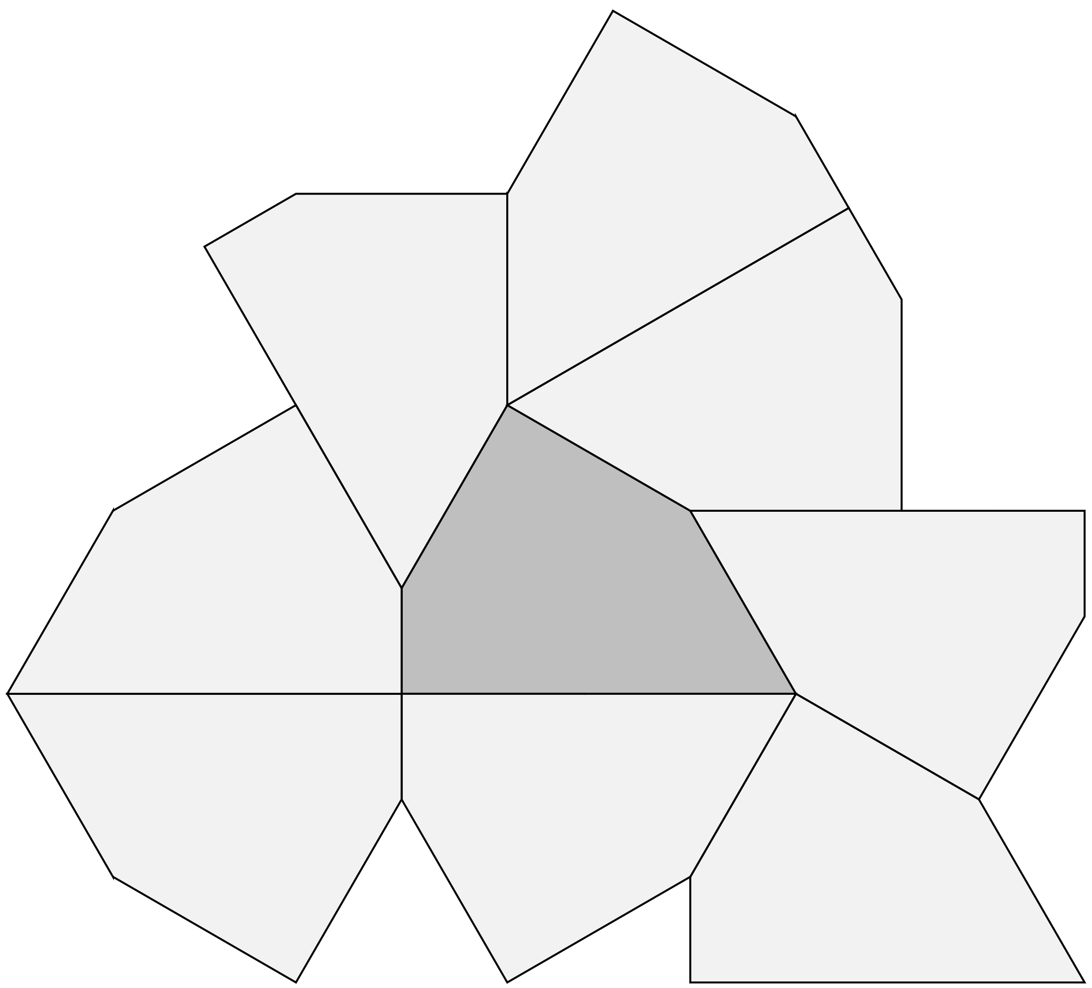
A **0-corona of S** is S itself, and a k-**corona of S** is a surround of a $(k-1)$-corona of S.

The **Heesch number** of S is the largest k for which S has a k-corona.

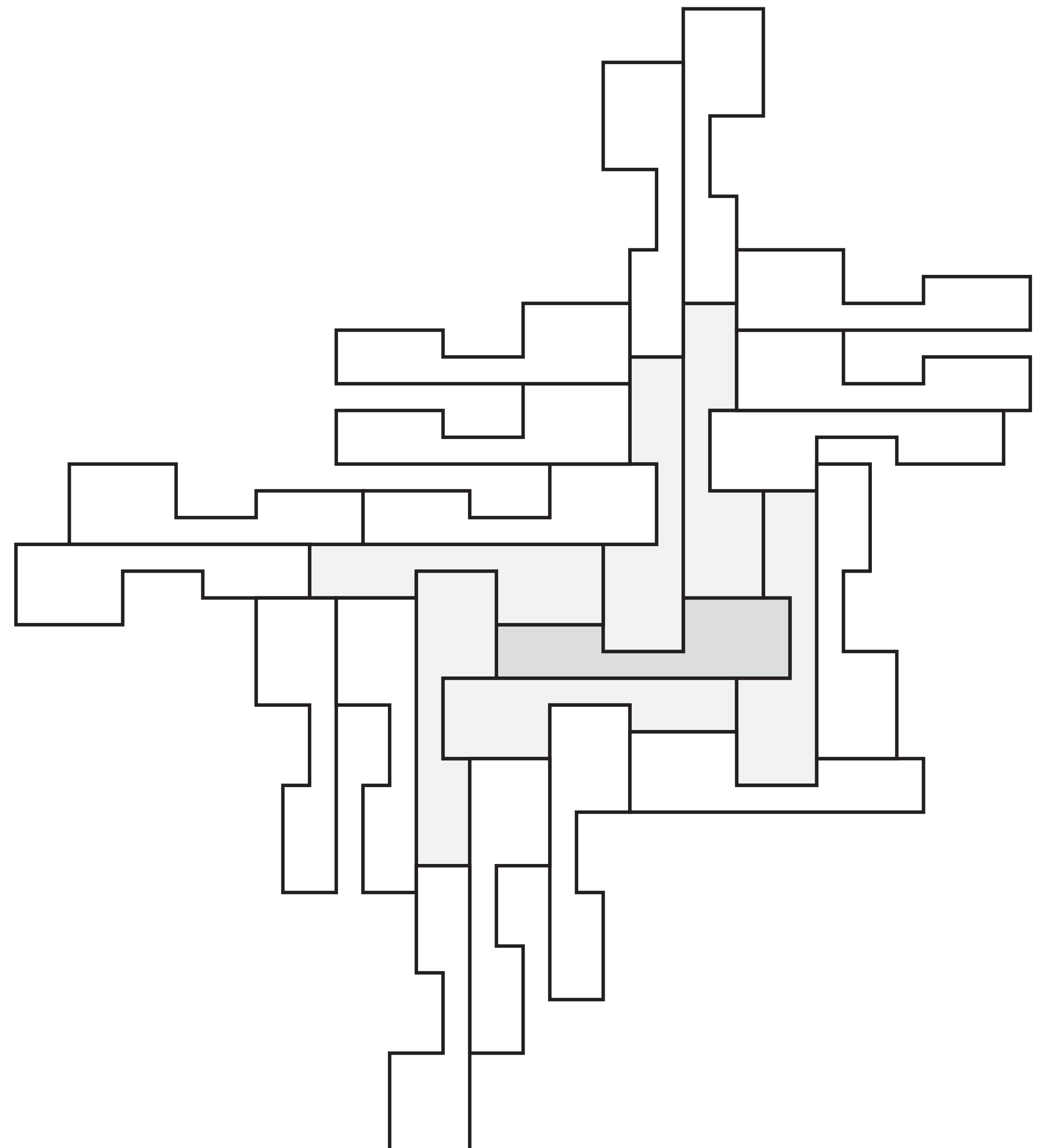
Heesch's Problem: for which positive integers n does there exist a shape with Heesch number n?



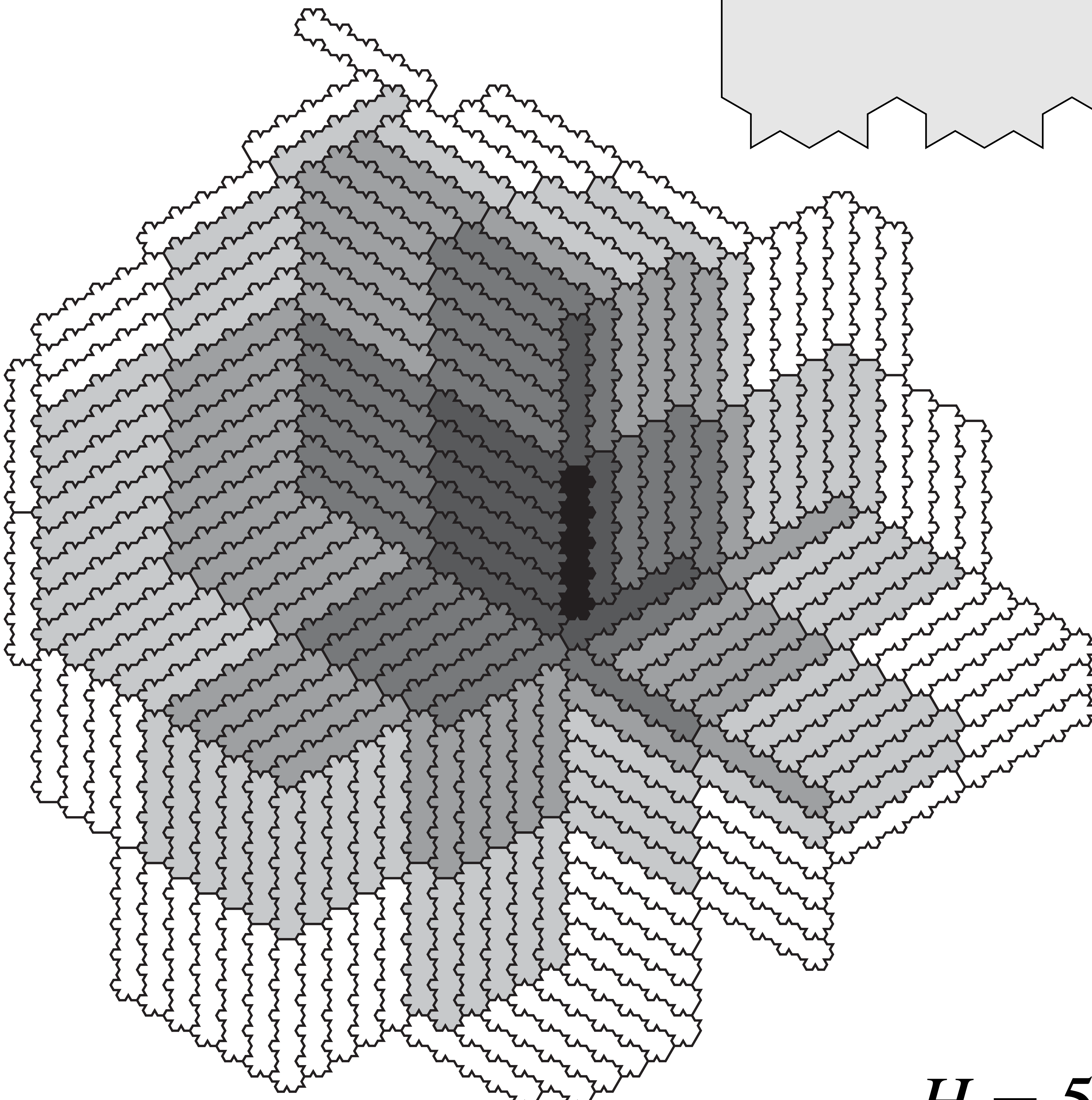
$H = 0$

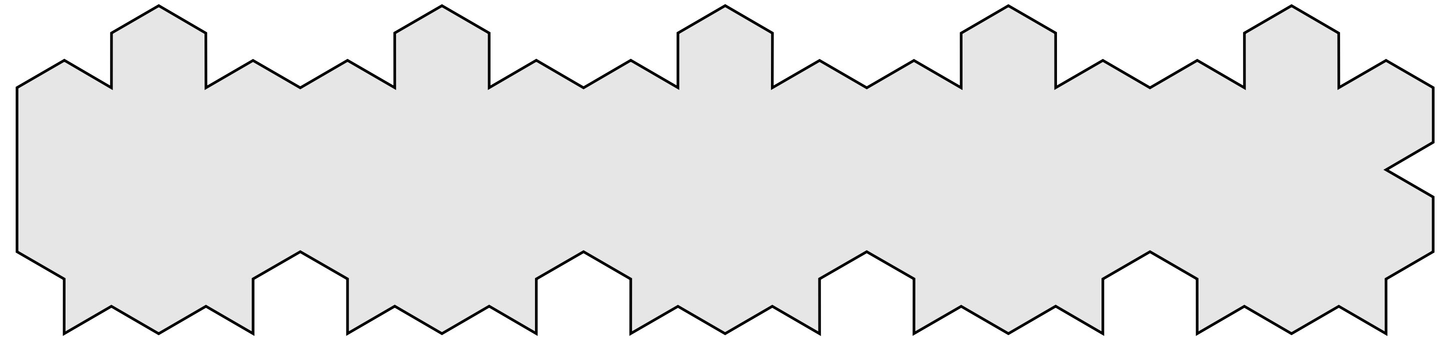


$H = 1$

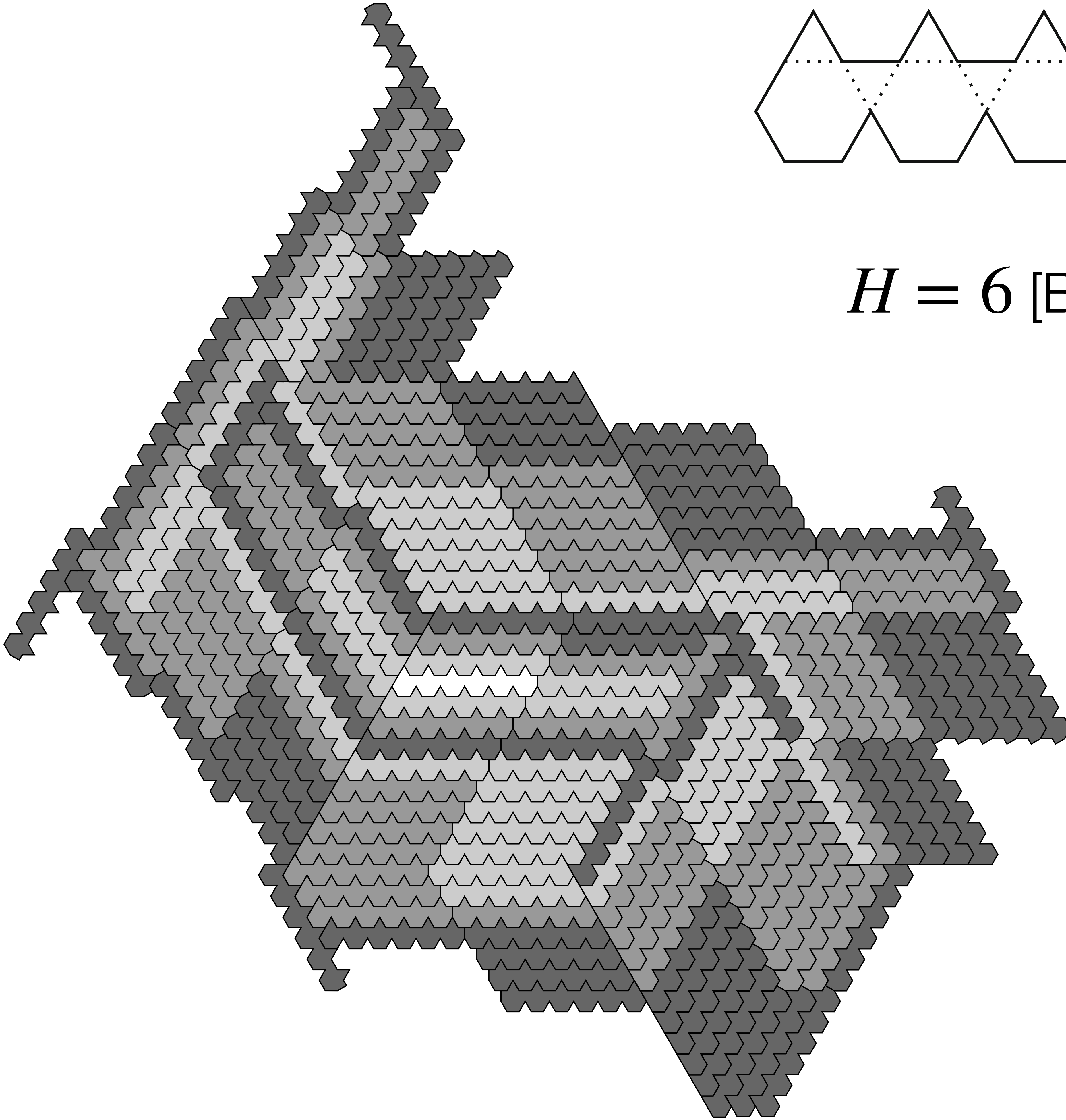


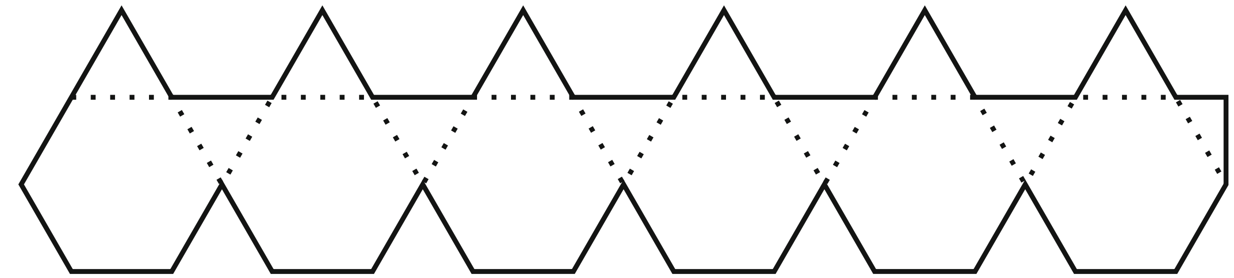
$H = 2$





$H = 5$ [Mann 2001]





$H = 6$ [Bašić 2021]

Computing Heesch numbers

Restrict attention to simple polyforms (e.g., polyominoes)

Assume (or prove?) that S doesn't tile the plane.

Compute all surrounds of S.

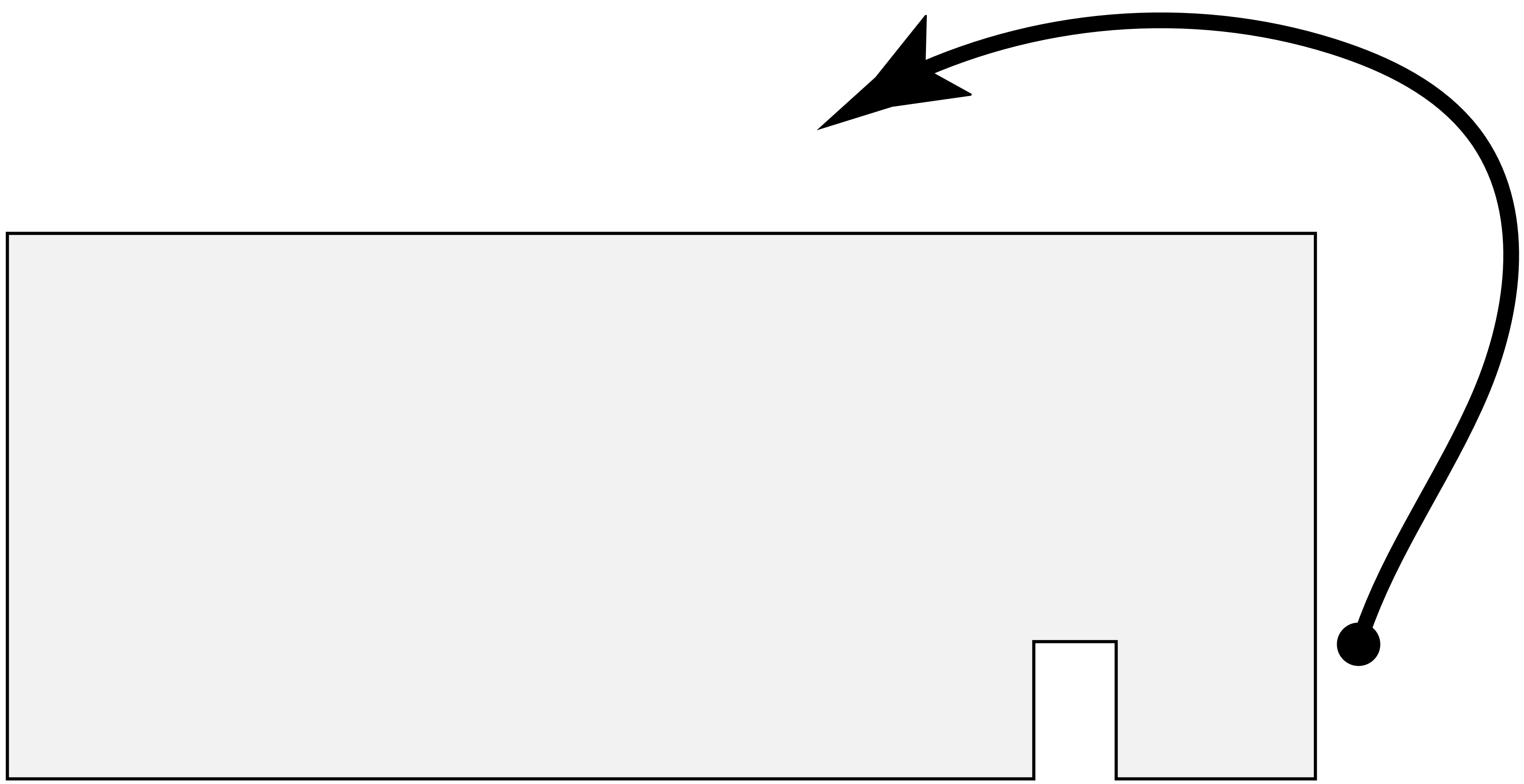
Compute all surrounds of the surrounds.

…

Report the largest number of layers you were able to generate in this enumeration.

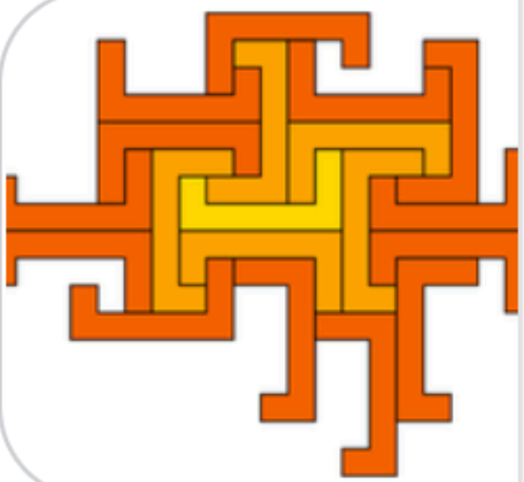
Backtracking hell

There could be many partial surrounds that all fail because of an incompatibility deep in the search.



March 2019

Craig S. Kaplan
See http://isohedral.ca/heesch-numbers-part-2-polyforms/. I've explored up to 14-ominoes, with a bunch of unclassified stuff from 11 onwards.



ISOHEDRAL.CA
Heesch Numbers, Part 2: Polyforms

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Bram Cohen
Craig S. Kaplan What are you using for your search? This seems like the sort of problem where a SAT solver will stomp any custom solver you write.

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Satisfiability (SAT)

Given a Boolean formula (using and, or, not, and any number of variables), is there an assignment of true or false to the variables that makes the formula true?

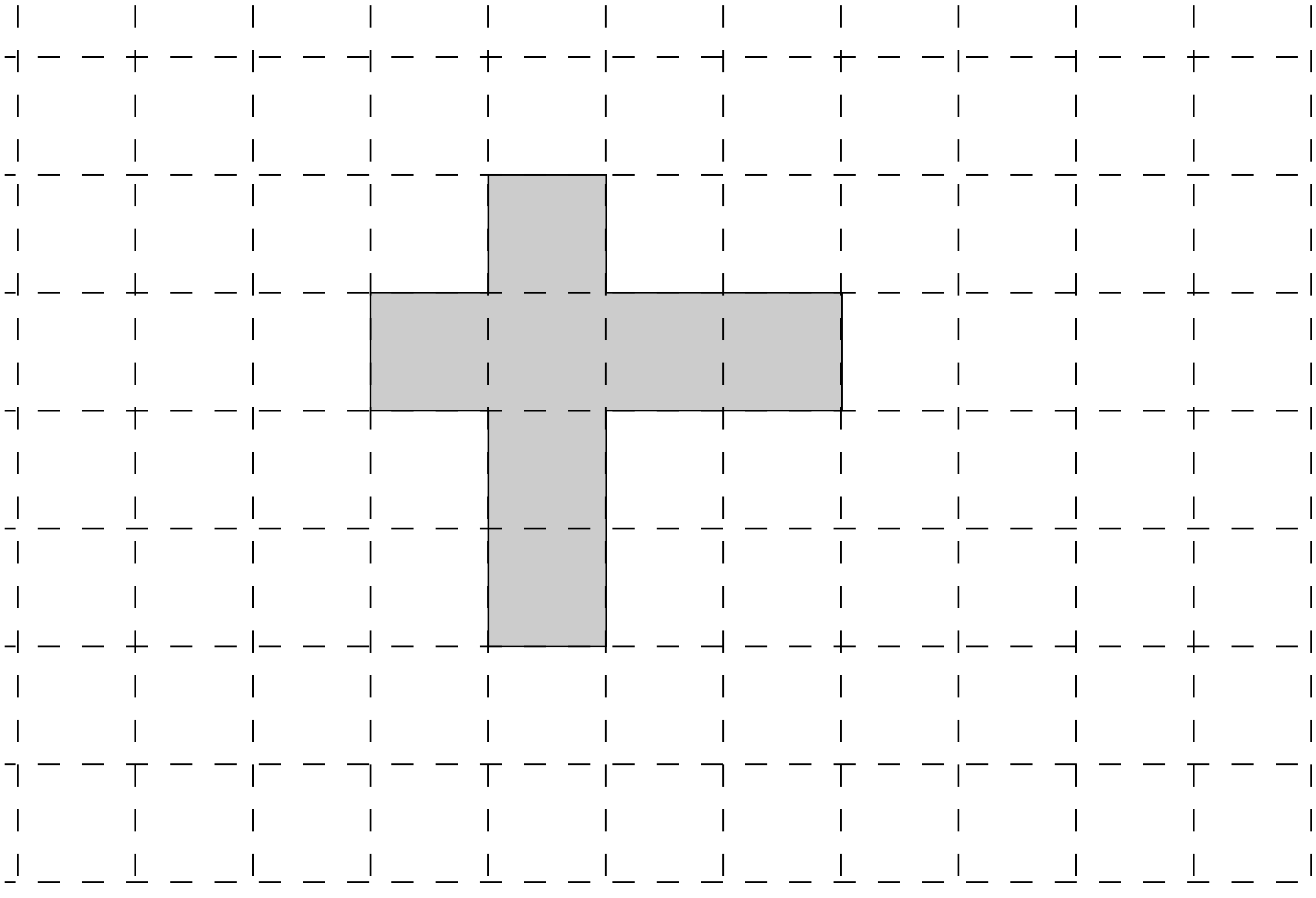
Conjunctive Normal Form (CNF)

A formula represented as an "and-of-ors"

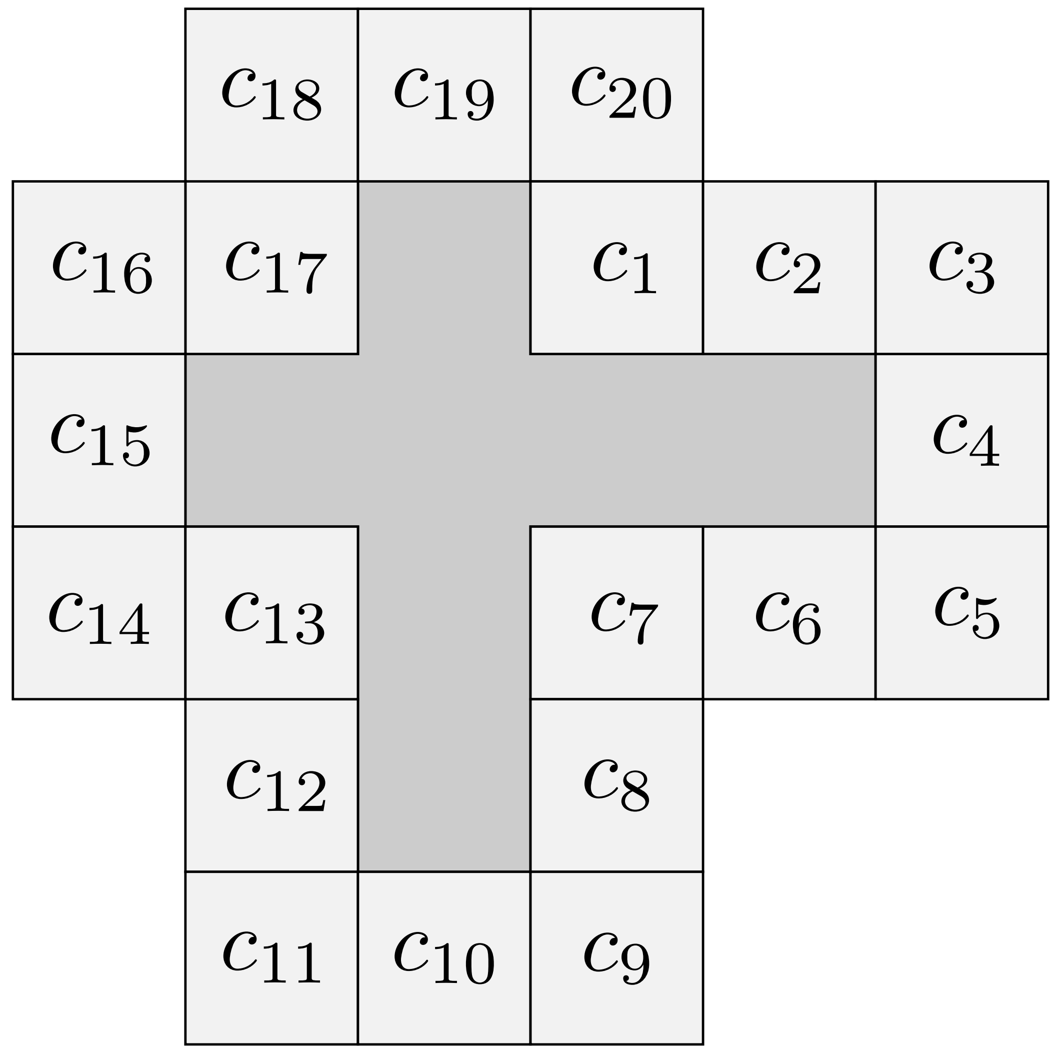
$$(A \vee \neg B \vee \neg C) \wedge (\neg D \vee E \vee F) \wedge (C)$$

or **not** **and**

Can a polyform be surrounded?

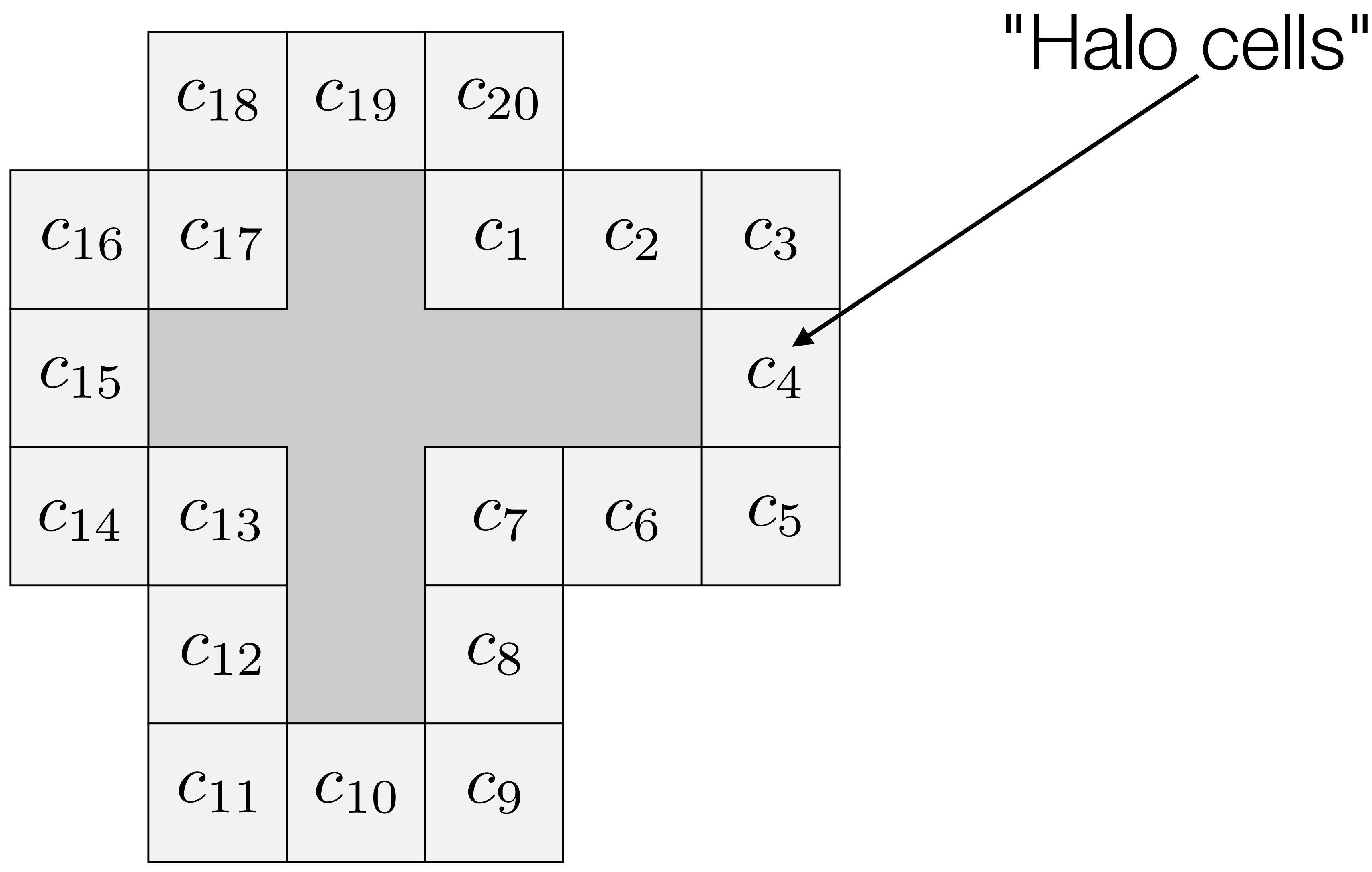


Cell Variables



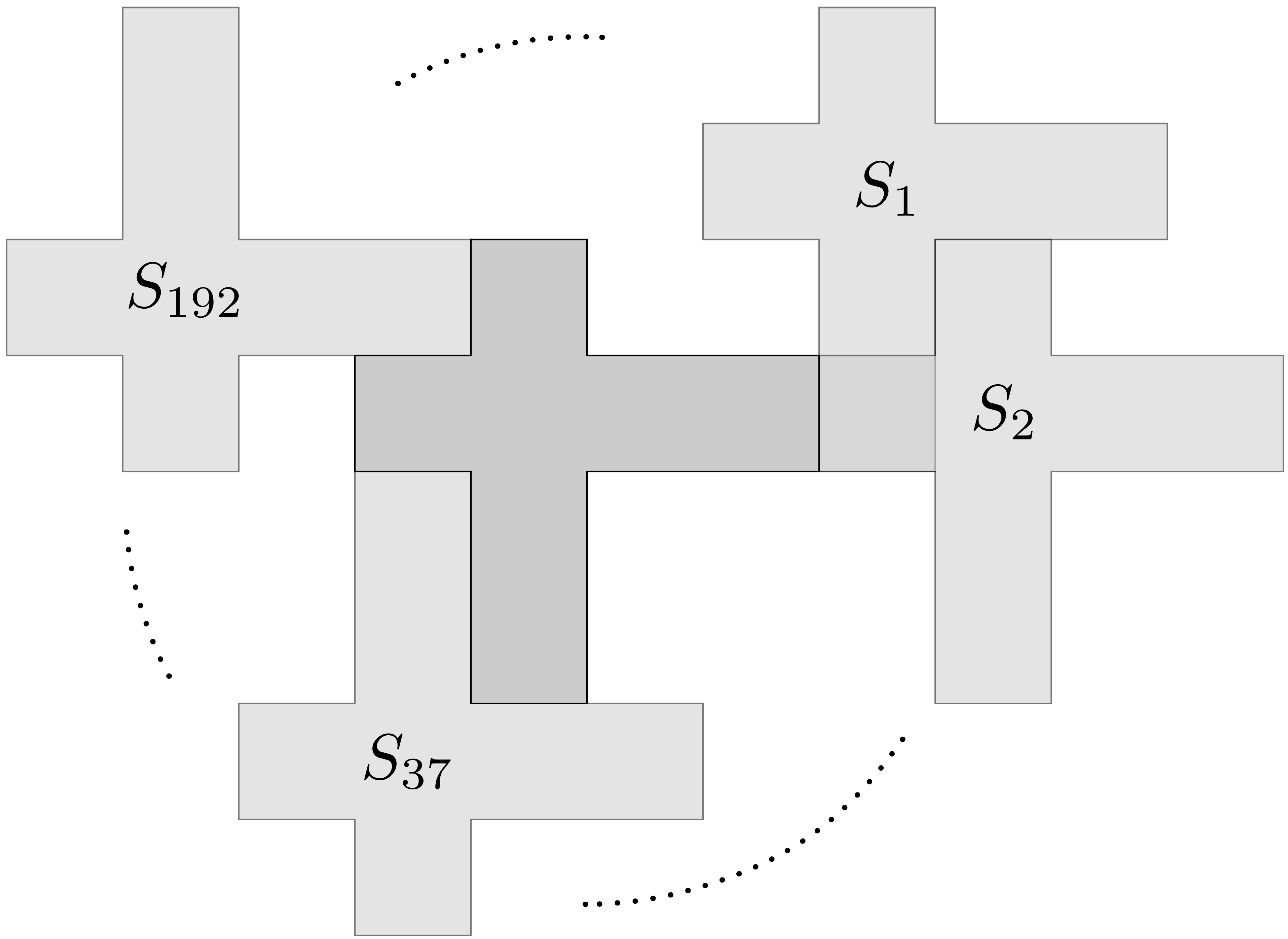
c_i is true if and only if the cell is occupied in a surround.

Cell Variables



c_i is true if and only if the cell is occupied in a surround.

Shape Variables



S_i is true if and only if the placement is used in the surround.

Clauses

$$(c_1) \wedge (c_2) \wedge \ldots \wedge (c_n)$$

Every cell in the shape's halo is occupied.

$$(\neg c_i \vee S_{k_1} \vee S_{k_2} \vee \ldots \vee S_{k_n})$$

$\forall S_{k_j}$ containing c_i

If a cell is occupied, some placement must use it.

$$\neg S_i \vee \neg S_j$$

$\forall S_i \cap S_j \neq \varnothing$

Overlapping placements can't both be used.

Checking for $H \geq n$

Cell variables

c_i is true if and only if the cell is used anywhere in a solution.

Shape variables

$S_{i,j}$ is true if and only if the ith placement is used as part of a j-corona.

Clauses

1. The 0-corona shape is used
2. A used placement occupies all of its cells
3. An occupied cell must be used by some placement
4. If two placements overlap, they cannot both be used

Clauses

1. The 0-corona shape is used
2. A used placement occupies all of its cells
3. An occupied cell must be used by some placement
4. If two placements overlap, they cannot both be used

5. Excluding the outermost corona (i.e., for $j < n$), If a placement is used, its halo cells must be occupied
6. If a placement is used in a j-corona, then some adjacent placement must be used in a $(j-1)$-corona
7. Adjacent placements whose coronas differ by more than 1 cannot both be used

Notes

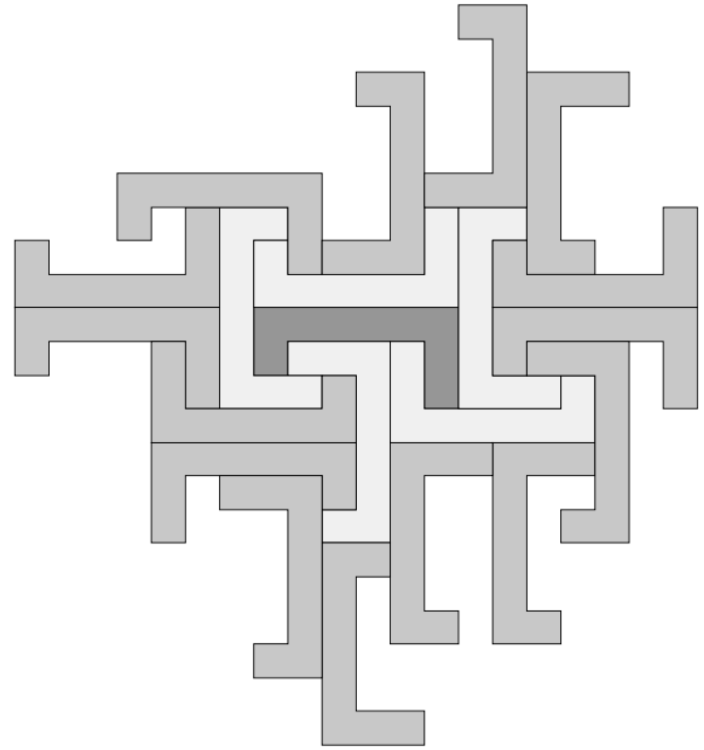
Need to know up front that the shape doesn't tile the plane

Build and run the SAT formula for progressively larger n until you fail

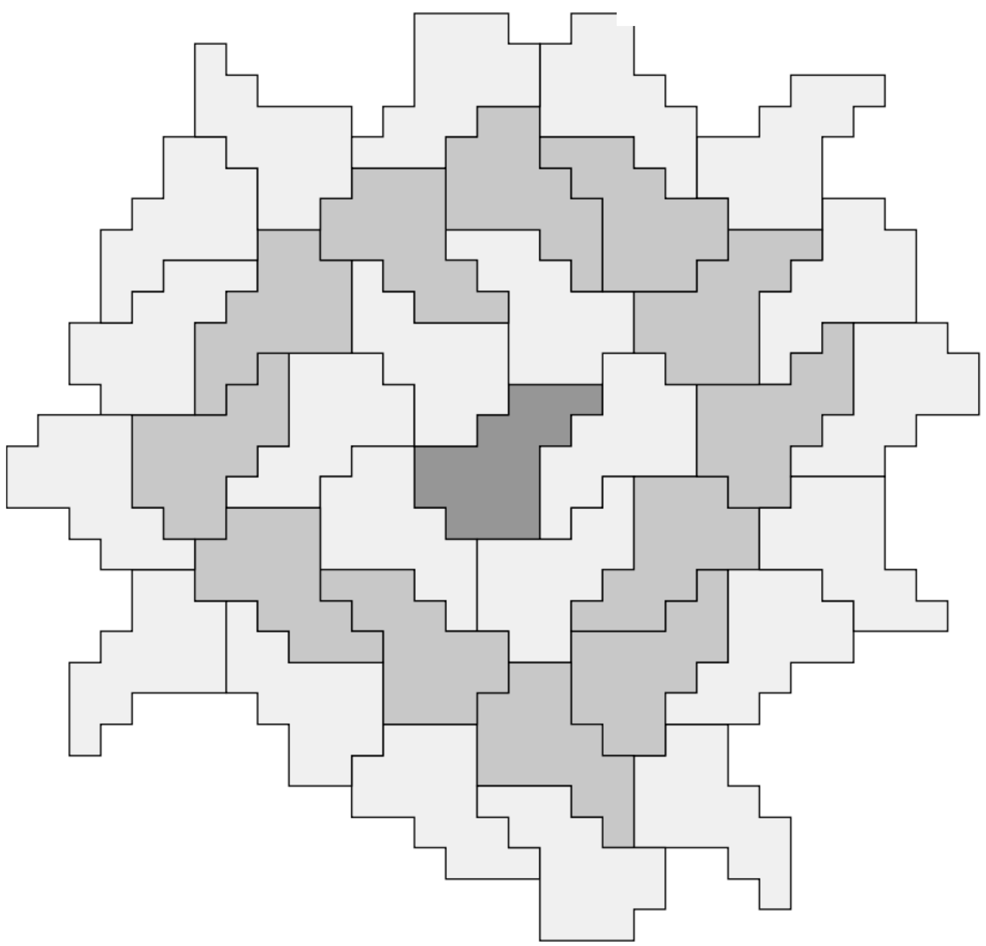
Need to do work outside the SAT solver to suppress holes in the outermost corona

Polyominoes

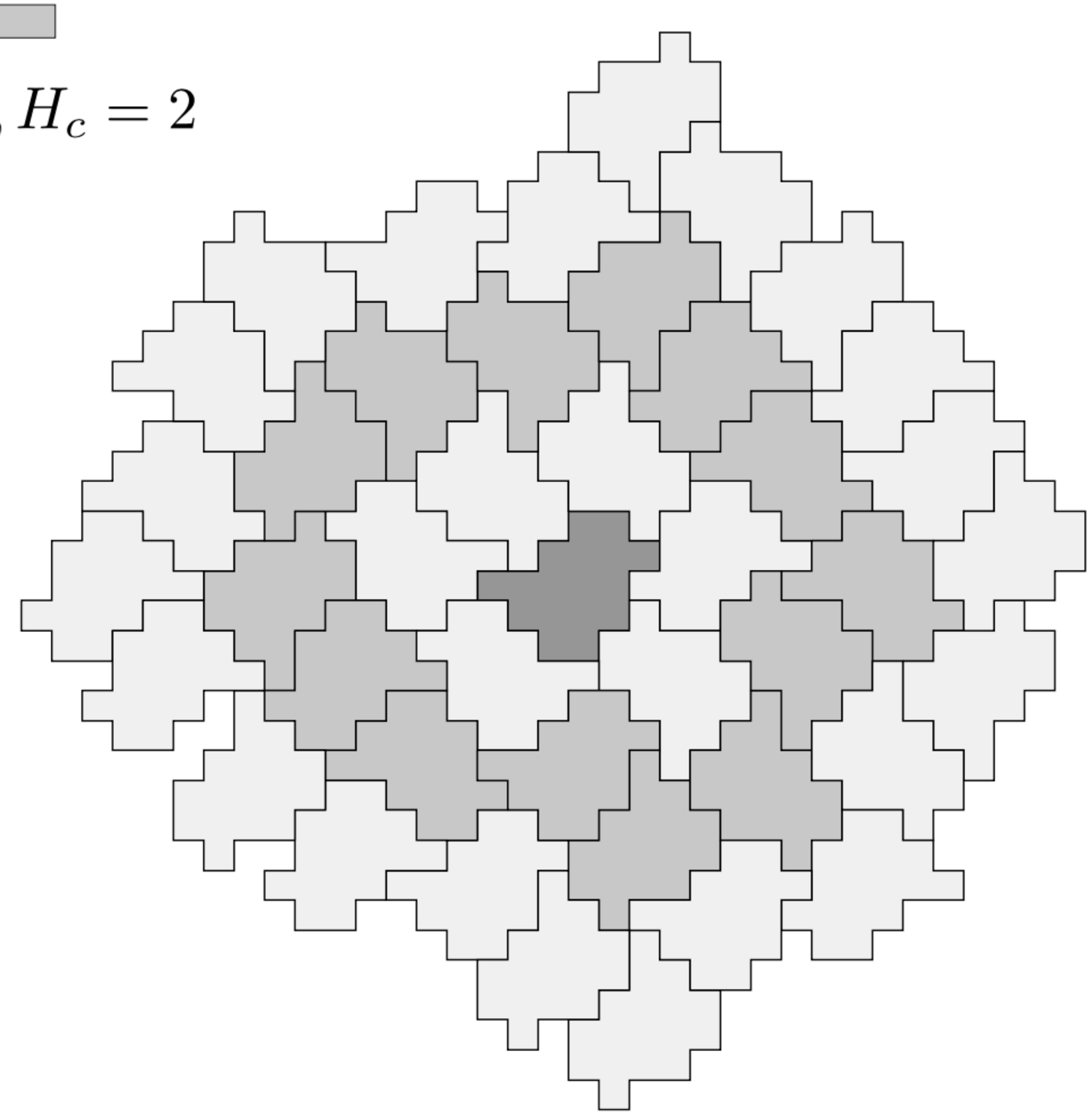
n	non-tilers	$H_c = 0$	$H_c = 1$	$H_c = 2$	$H_c = 3$
7	3	1	2		
8	20	6	14		
9	198	75	122	1	
10	1390	747	642	1	
11	9474	5807	3628	39	
12	35488	28572	6906	10	
13	178448	149687	28694	67	
14	696371	635951	60362	58	
15	2721544	2598257	123262	25	
16	10683110	10397466	285578	66	
17	41334494	40695200	639162	130	2
18	155723774	154744331	979375	68	
19	596182769	593856697	2325874	198	



9-omino, $H_c = 2$



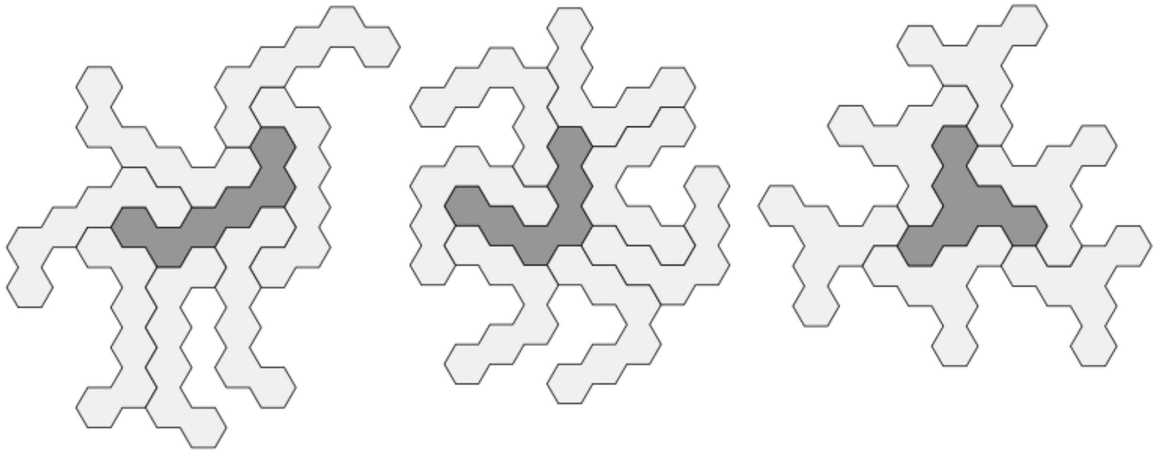
17-omino, $H_c = 3$



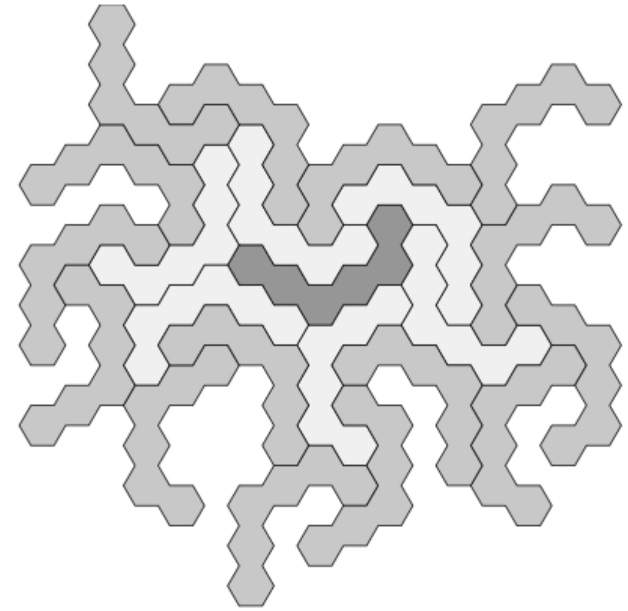
17-omino, $H_c = 3$

Polyhexes

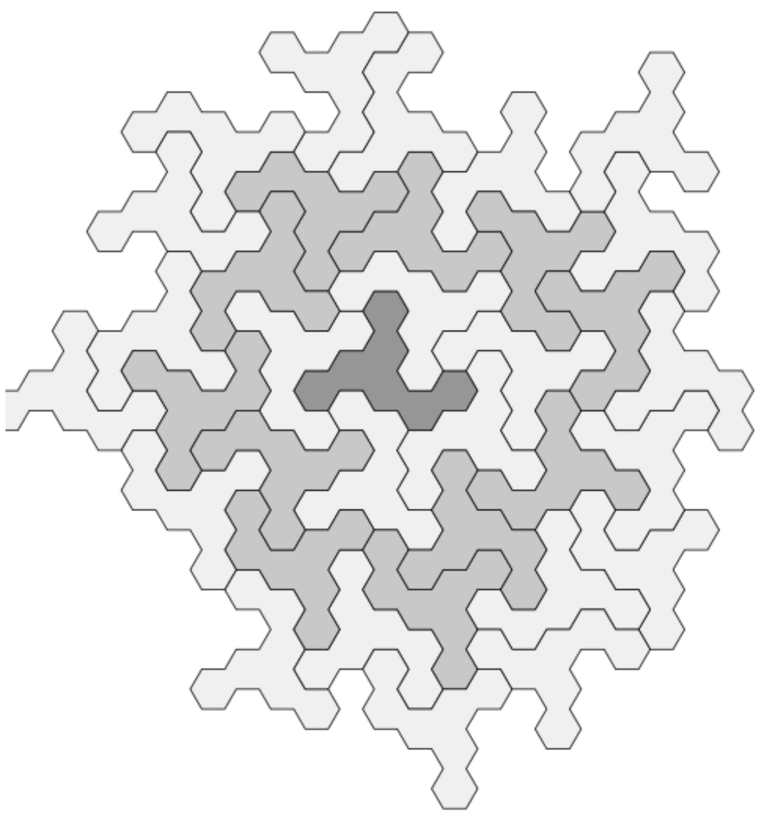
n	non-tilers	$H_c = 0$	$H_c = 1$	$H_c = 2$	$H_c = 3$	$H_c = 4$
6	4		3	1		
7	37	5	25	6	1	
8	381	70	264	44	3	
9	2717	825	1822	67	3	
10	18760	8248	10234	265	13	
11	116439	67644	47940	817	37	1
12	565943	431882	133484	567	10	
13	3033697	2565727	466159	1783	27	1
14	14835067	13676416	1156793	1836	22	
15	72633658	69871458	2758485	3534	179	2
16	356923880	350337478	6581529	4818	54	1
17	1746833634	1731652467	15167876	13129	161	1



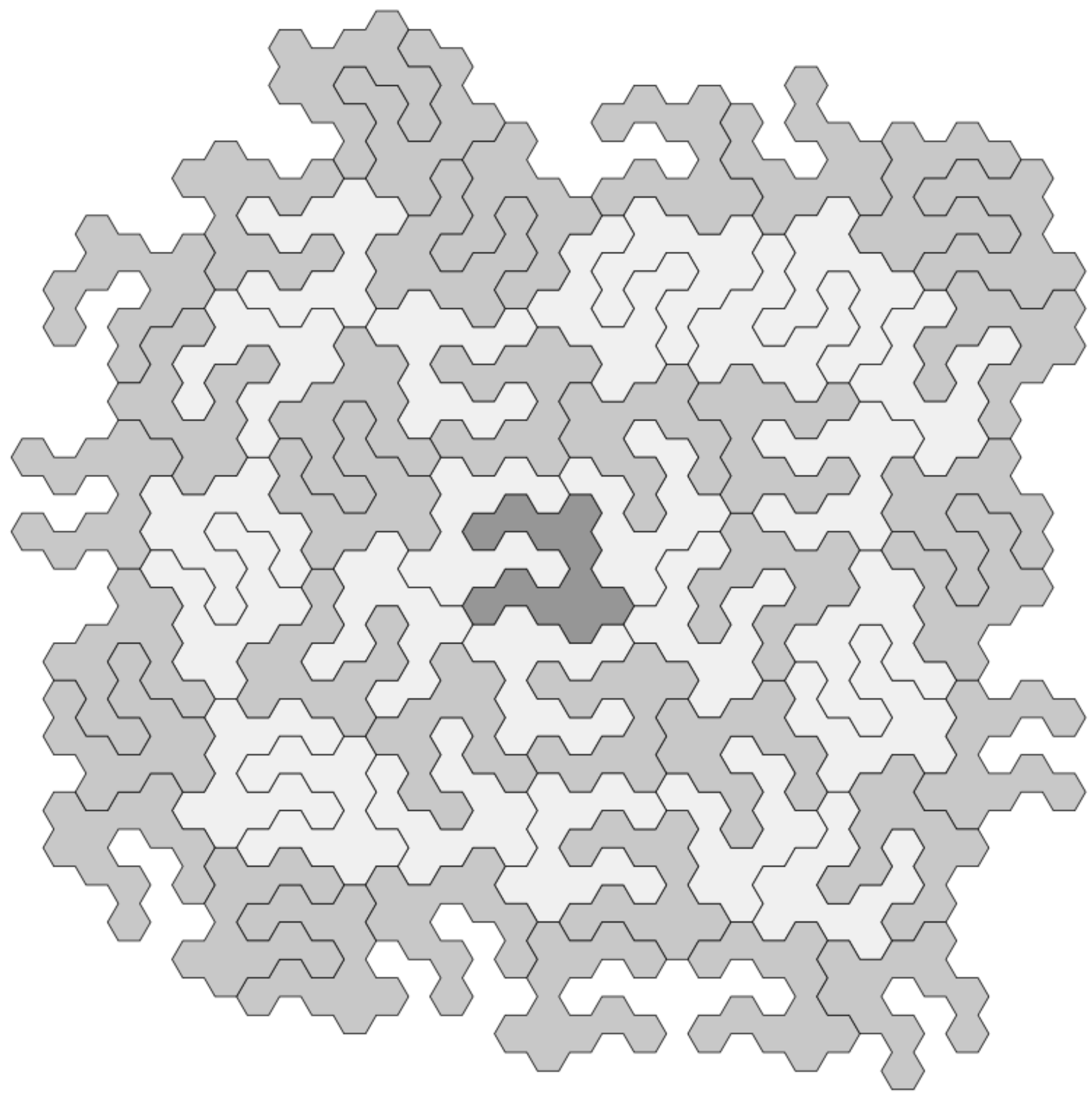
6-hexes, $H_c = 1$



6-hex, $H_c = 2$



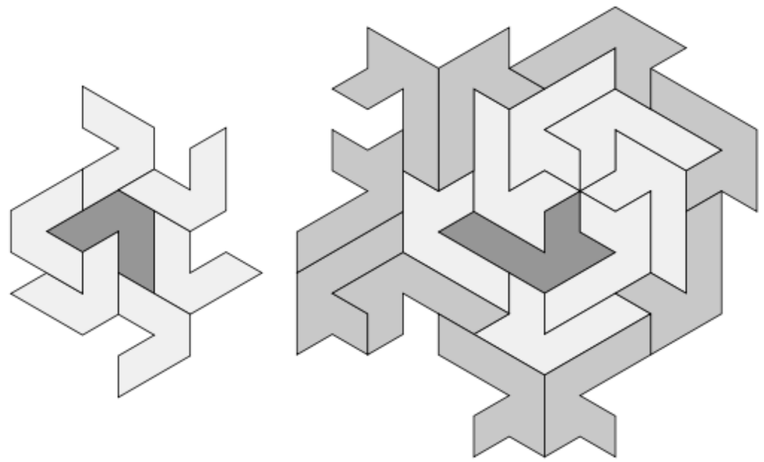
7-hex, $H_c = 3$



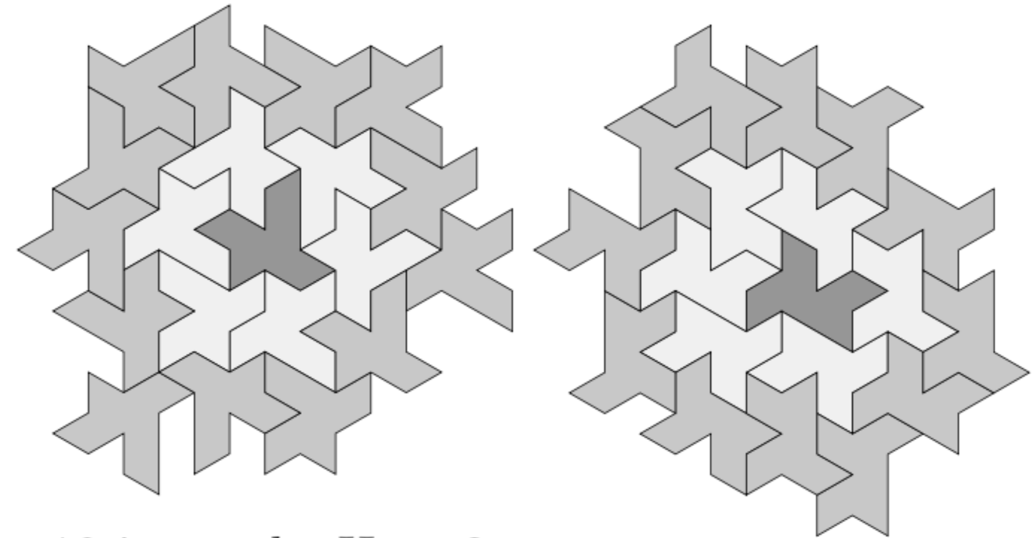
11-hex, $H_c = 4$

Polyiamonds

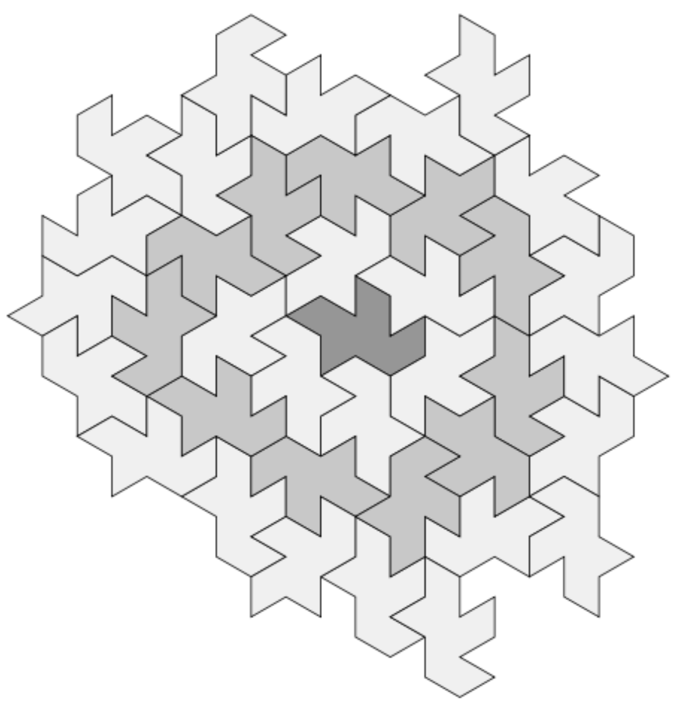
n	non-tilers	$H_c = 0$	$H_c = 1$	$H_c = 2$	$H_c = 3$	$H_c = 4$
7	1		1			
8	0					
9	20	11	9			
10	103	44	55	3	1	
11	594	236	346	11	1	
12	1192	826	364	1	1	
13	6290	4360	1884	24	2	
14	18099	14949	3141	8		
15	54808	48108	6661	39		
16	159048	148881	10153	13	1	
17	502366	474738	27544	83	1	
18	1374593	1341460	33100	33		
19	4076218	4001470	74689	57	2	
20	11378831	11282686	96091	51	2	1
21	32674779	32505745	168959	73	2	
22	93006494	92740453	265977	62	2	
23	264720498	264216706	503651	140	1	
24	748062099	747476118	585571	384	26	



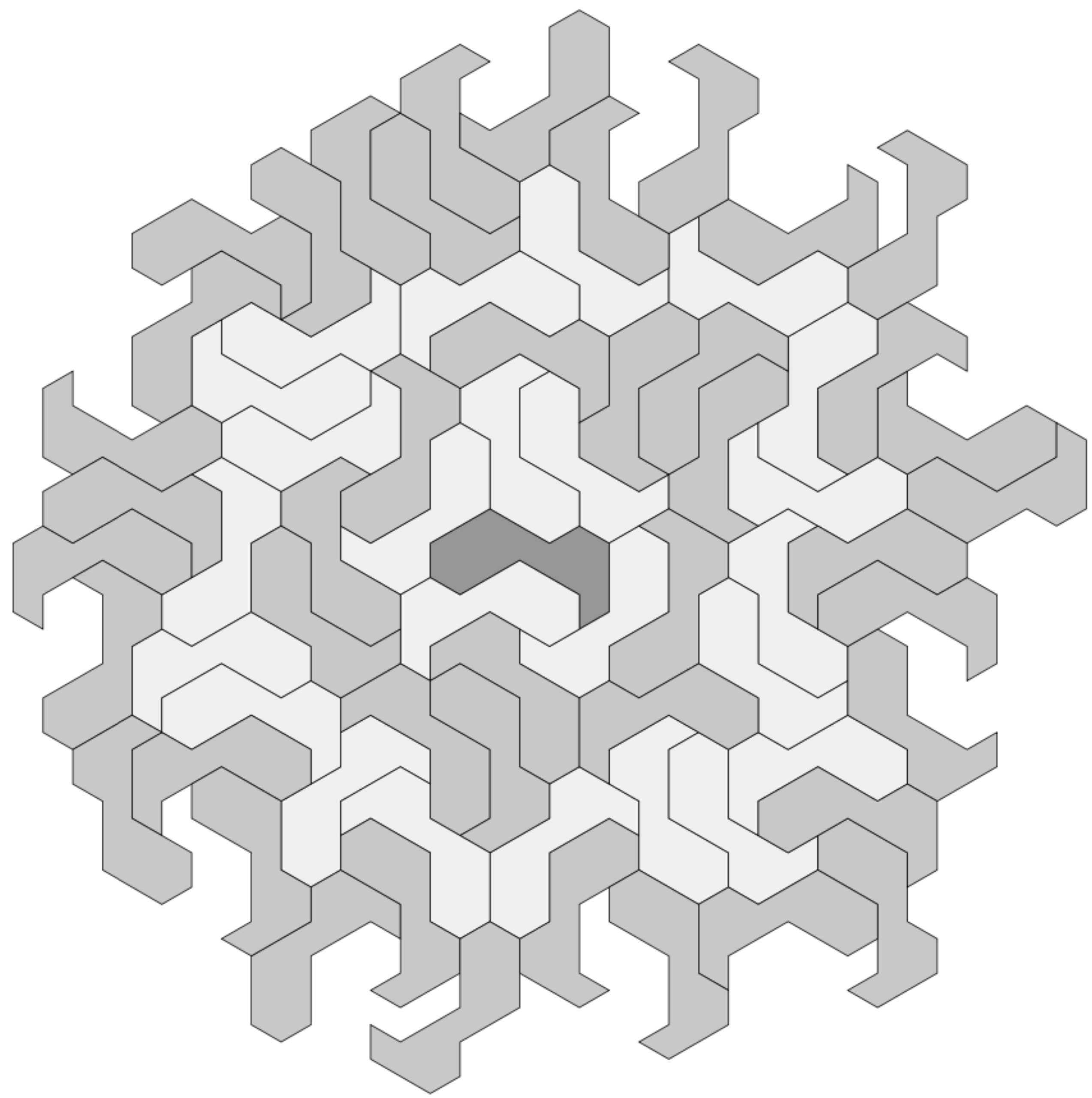
7-iamond, $H_c = 1$



10-iamonds, $H_c = 2$



10-iamond, $H_c = 3$



20-iamond, $H_c = 4$

Future Work

1. Keep going: more polyominoes, more polyhexes, more polyiamonds

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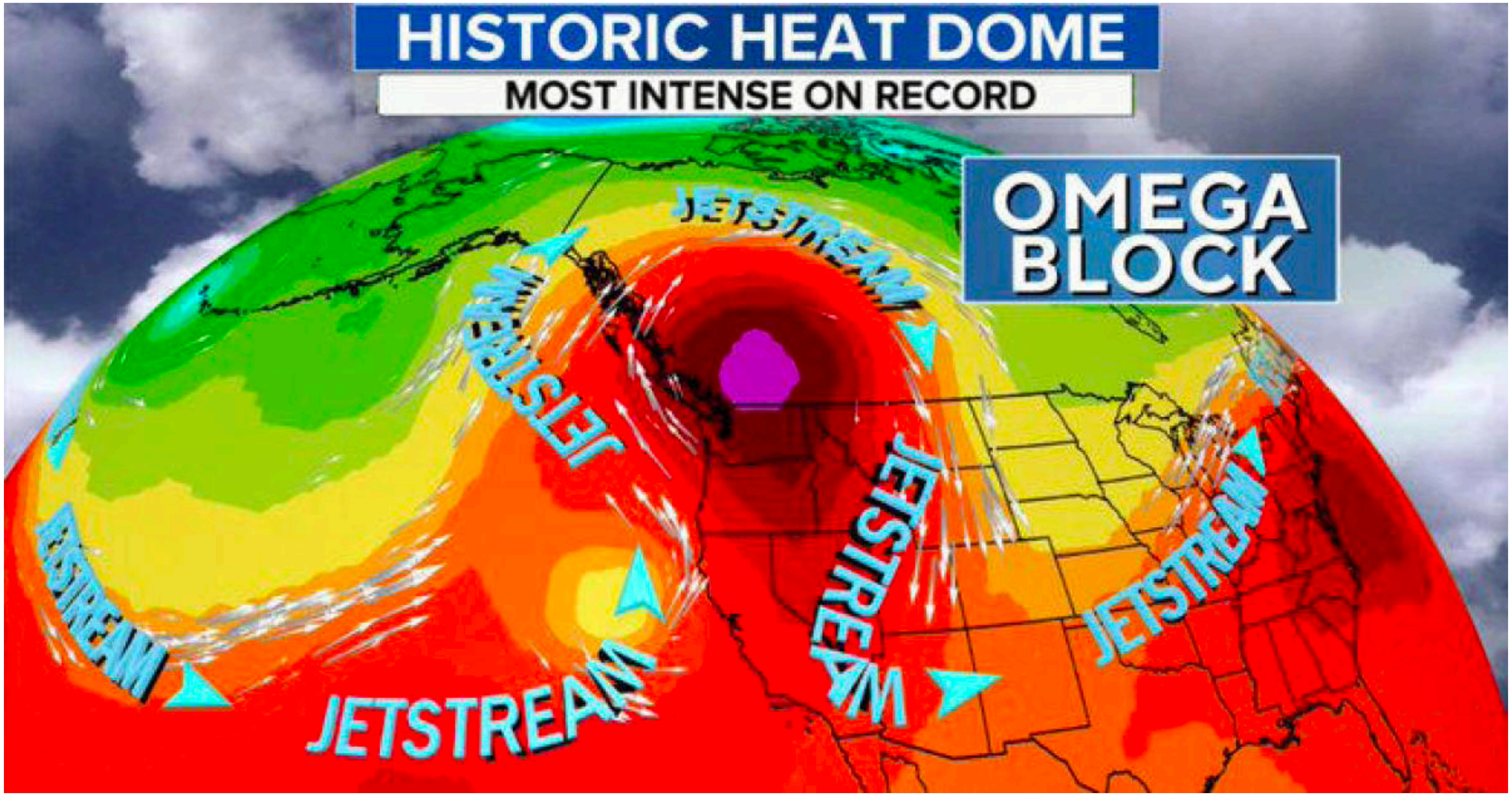
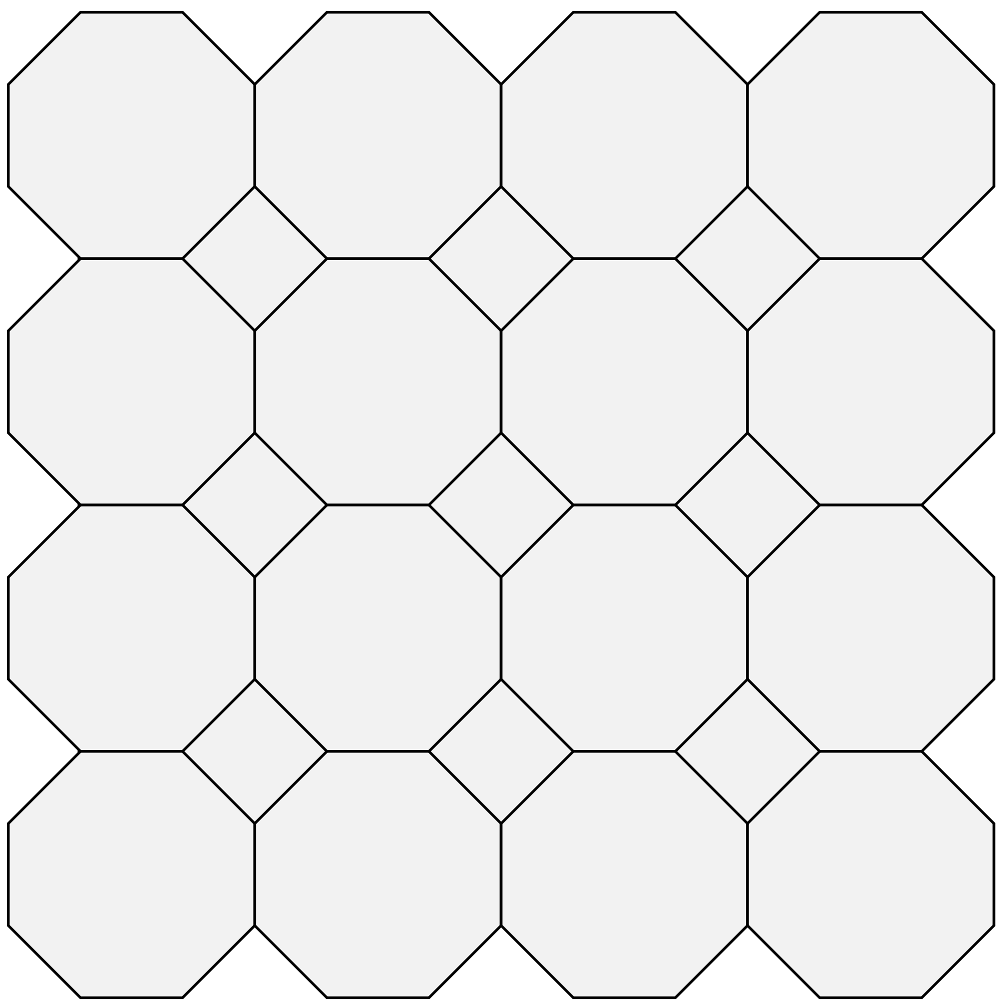
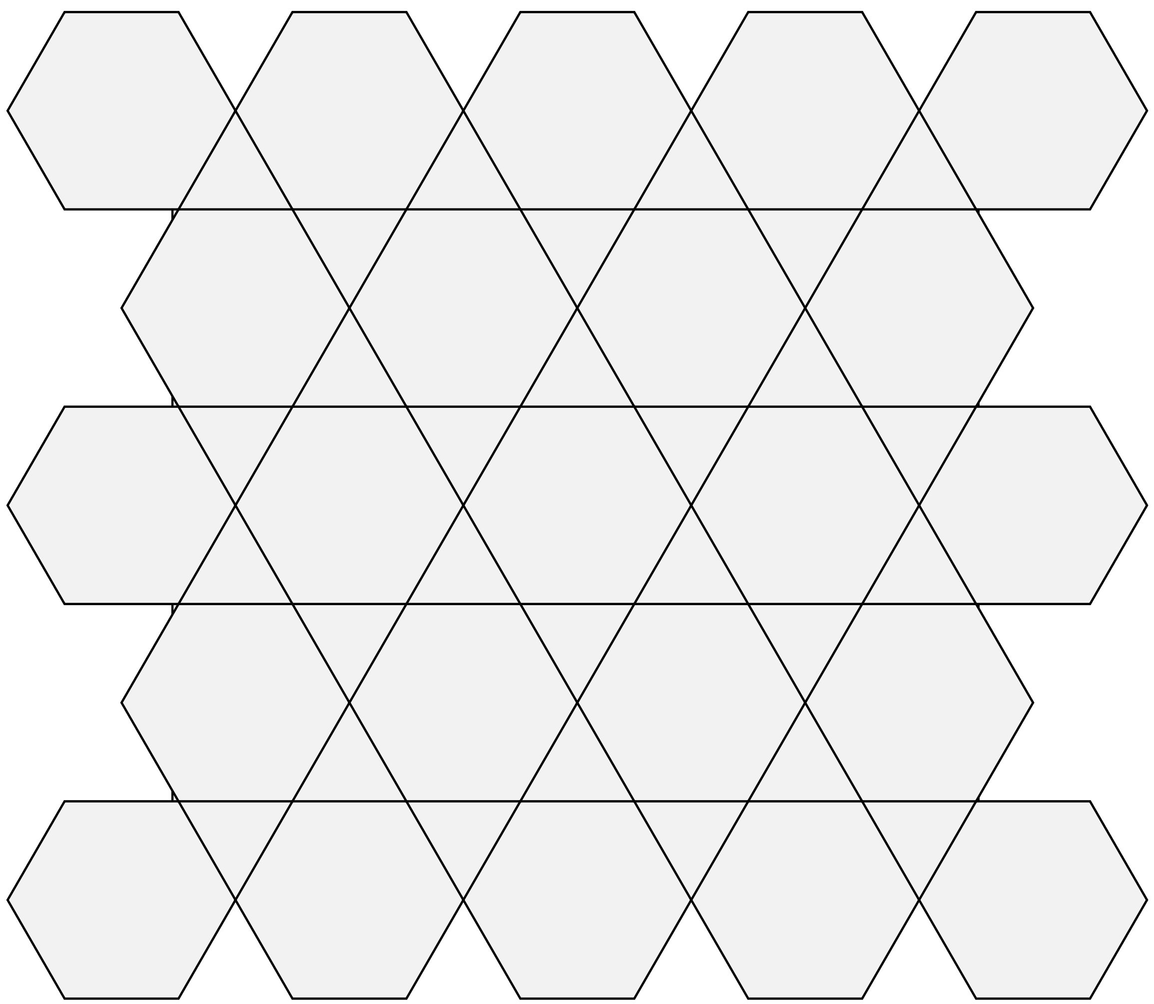


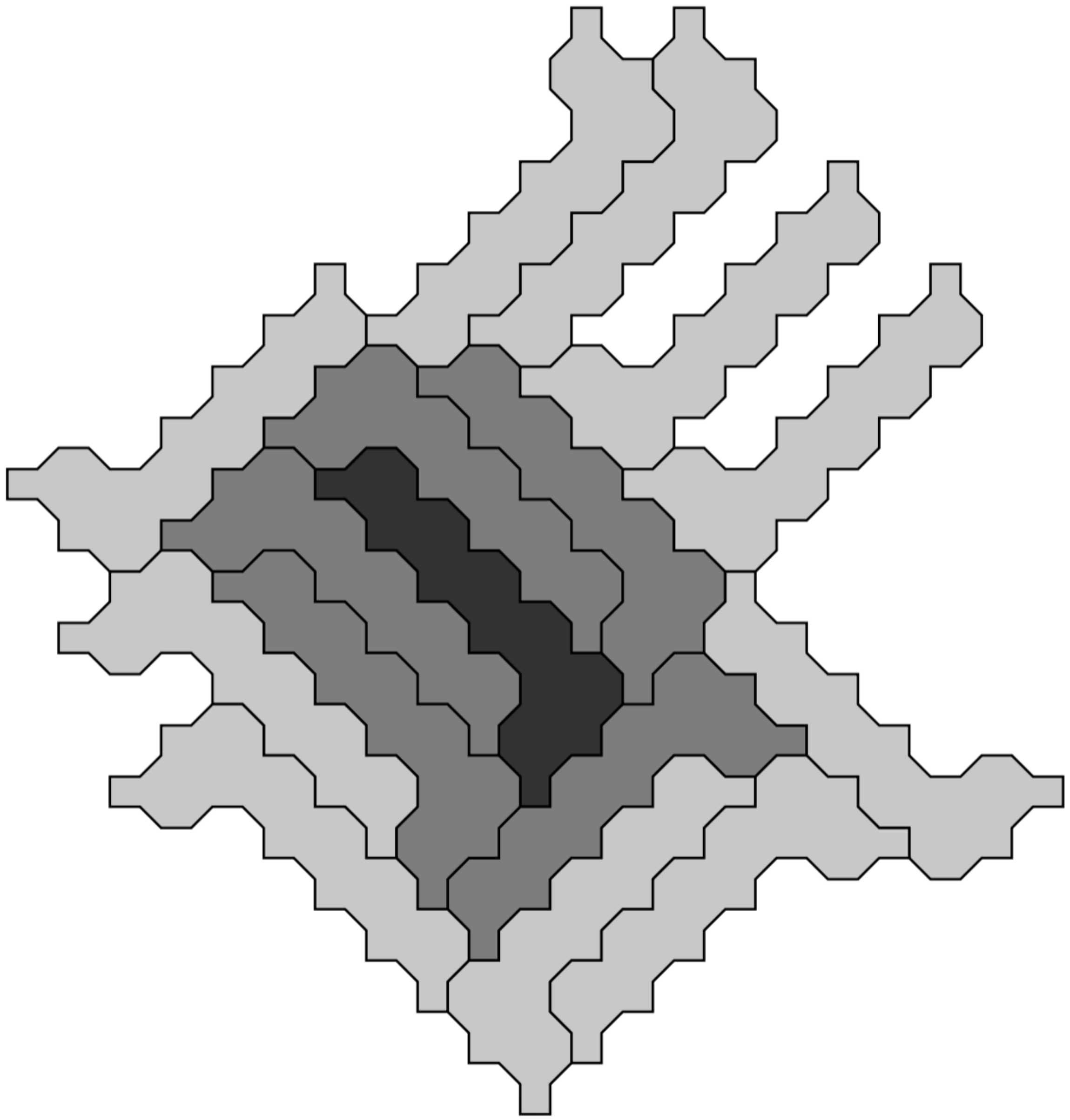
Tile smarter, not harder!

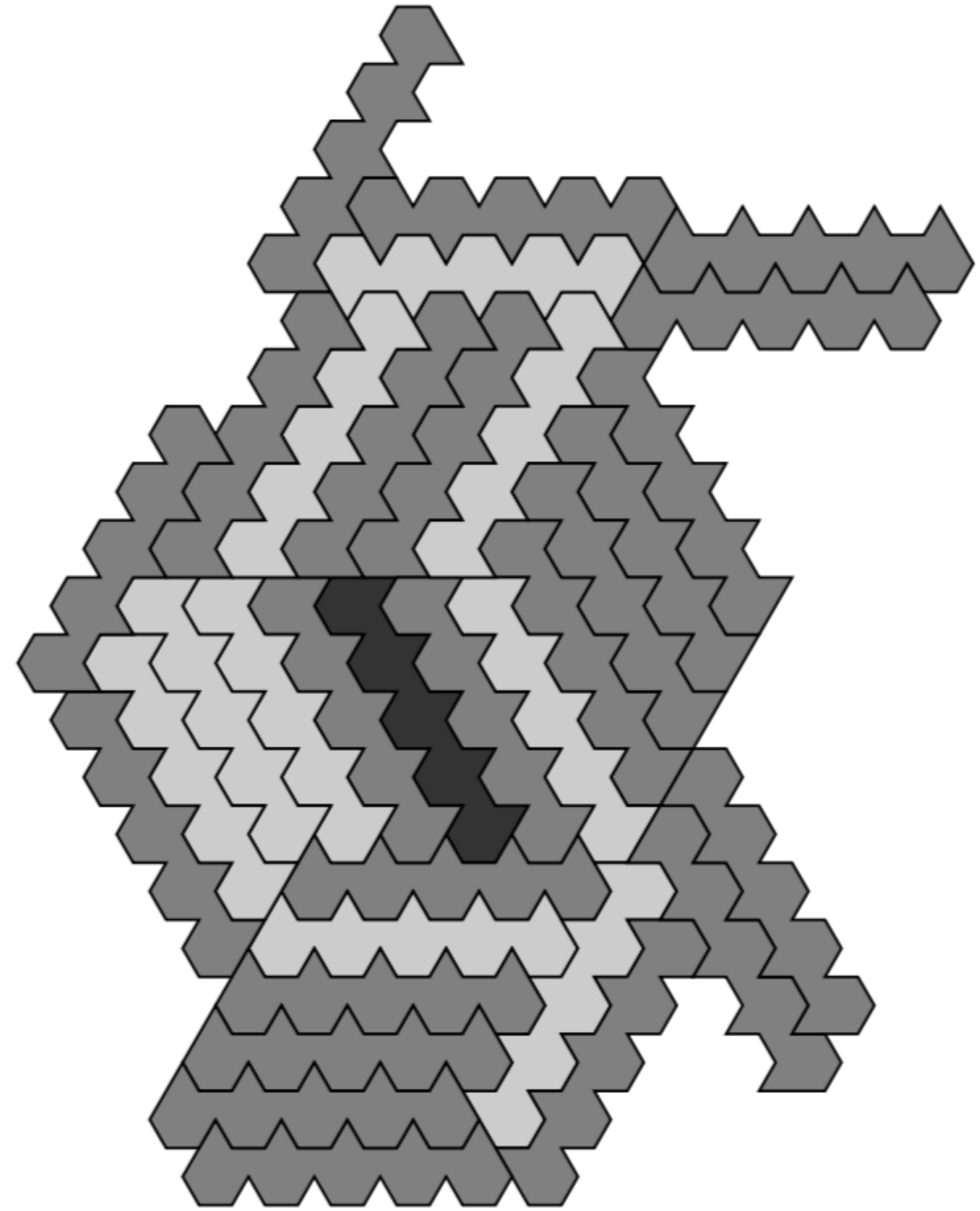
Future Work

2. Other ambient grids

Convenient tricks to make sure you generate only non-tiling shapes.

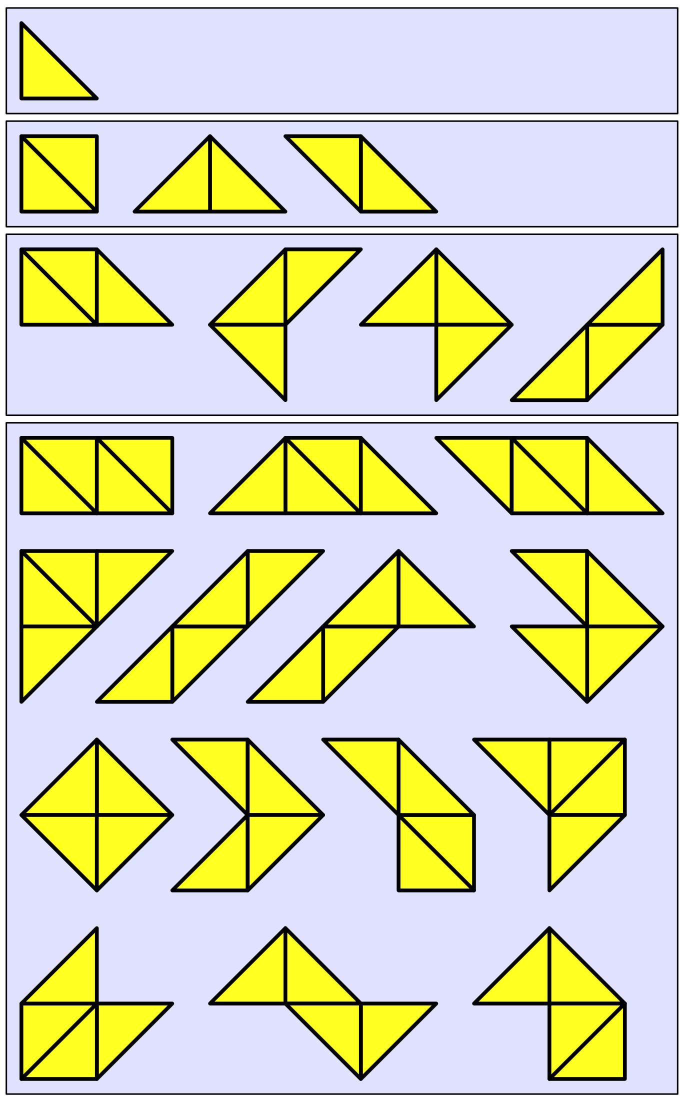




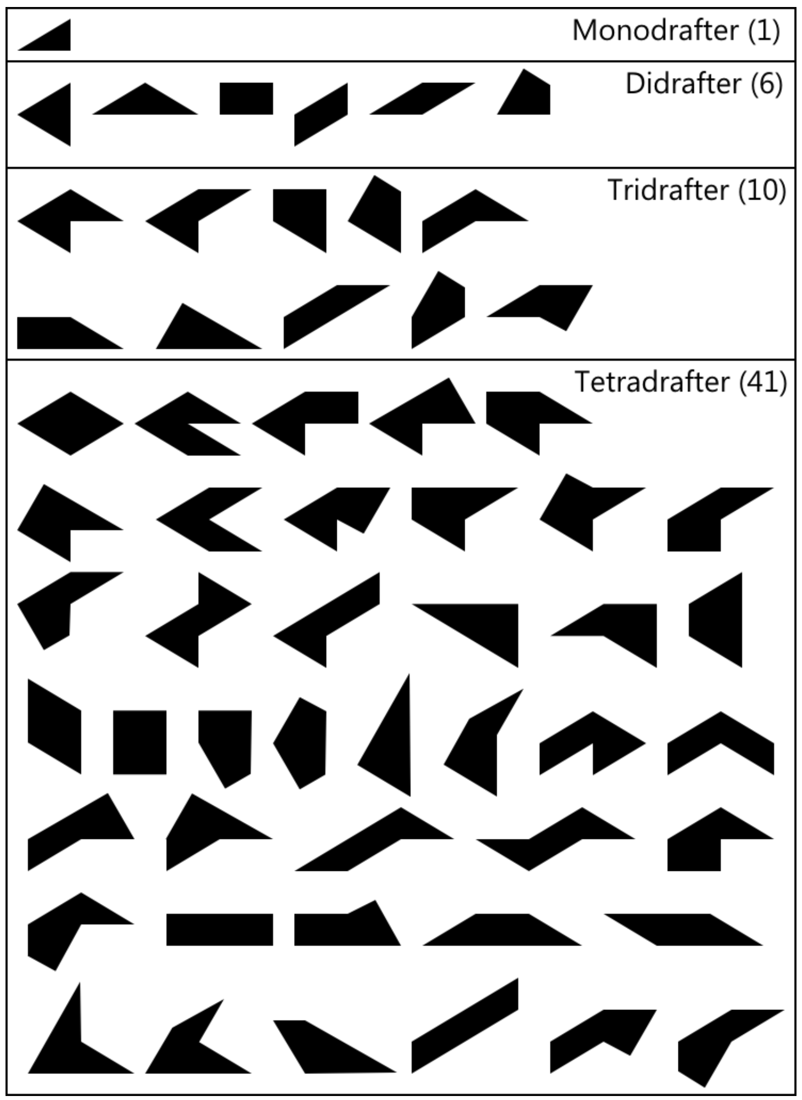
Joint work with Ava Pun, University of Waterloo

Future Work

2. Less structured forms



Polyabolos



Polydrafters

Eliminate shapes that tile, or just barrel through them

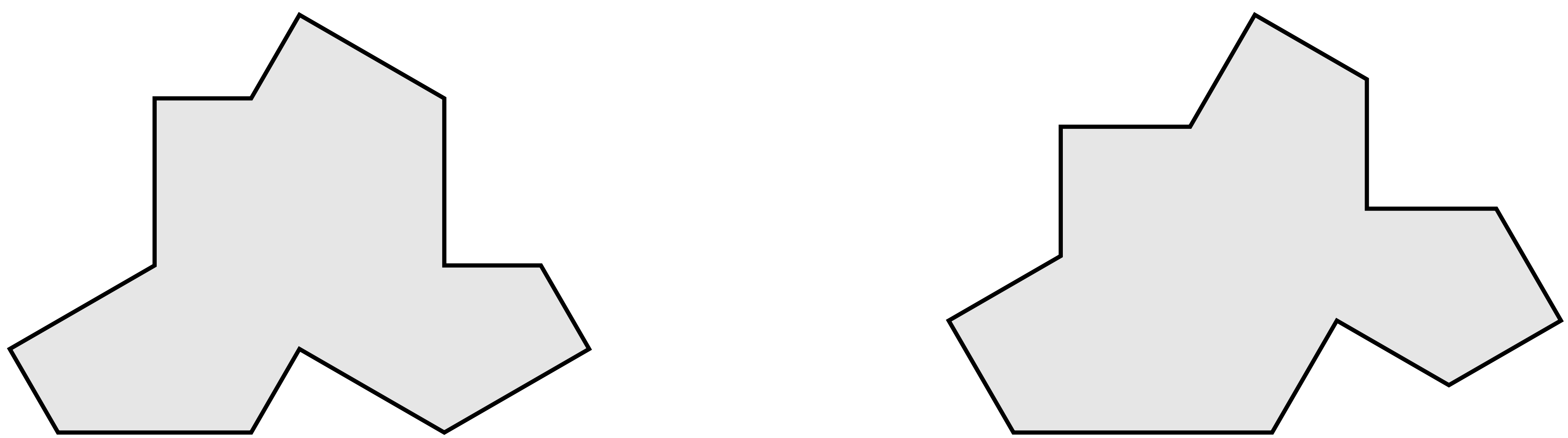
High Heesch numbers

Finding patterns of shapes with high Heesch numbers could point the way towards shapes that tile aperiodically…

High Heesch numbers

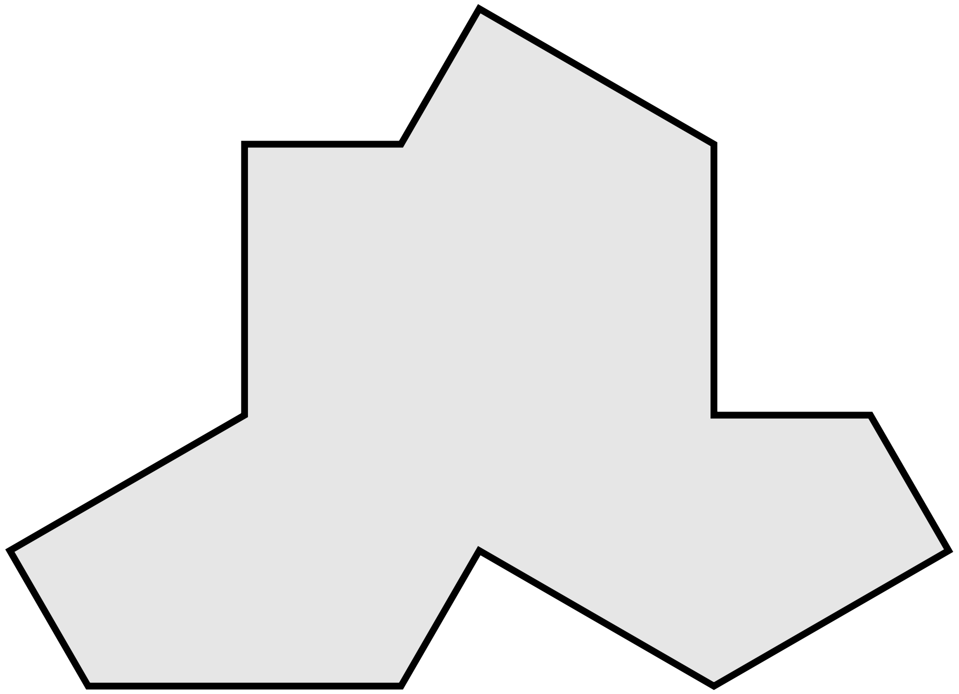
Finding patterns of shapes with high Heesch numbers could point the way towards shapes that tile aperiodically…

…or just wait for David Smith to send you the answer.

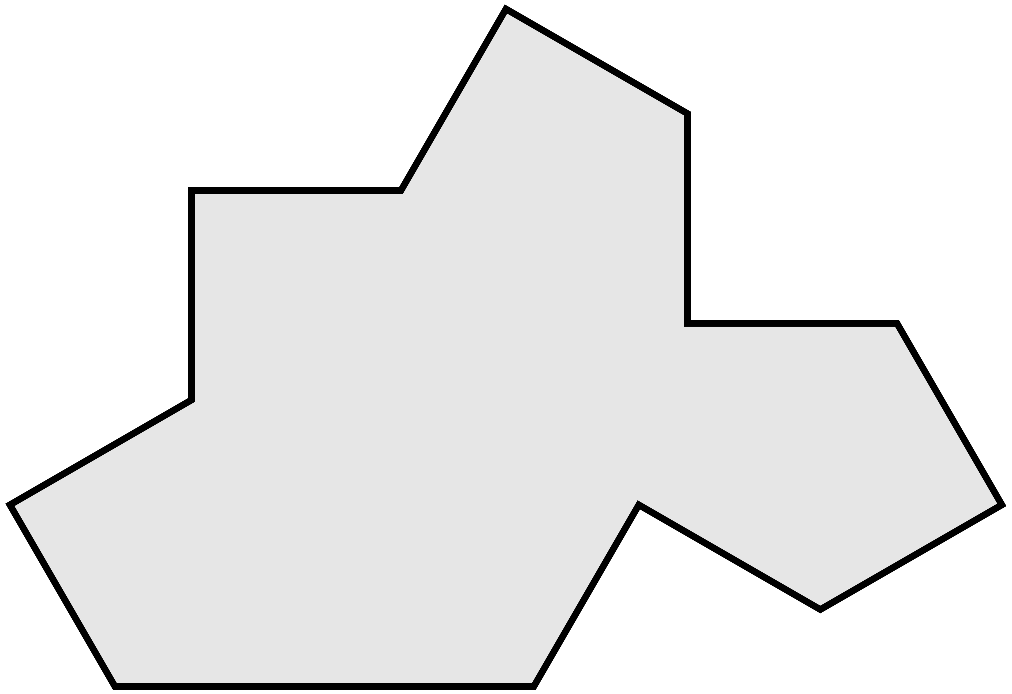


Aperiodic monotiles

Ask me about aperiodic monotiles!



The Hat, March 2023



The Spectre, May 2023

Thank you!

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