Proving the Prime Number Theorem *in an hour*

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Abstract

A proof of the prime number theorem is presented, using nothing more advanced than the residue theorem from complex analysis. Based on the paper *Newman's Short Proof of the Prime Number Theorem* by Don Zagier.

The statement

- Let $\pi(x)$ count the primes $\leqslant x$, i.e., $\pi(x) \coloneqq \sum_{p \leqslant x} 1$.
- The *Prime Number Theorem* states that

$$\lim_{x \to \infty} \left(\pi(x) \Big/ \frac{x}{\ln x} \right) = 1.$$

- The simplest proofs require the use of complex numbers!

Notation

- Throughout let $z = x + iy$ be a complex number with x, y real, and let p be prime.

Analytic functions

- Complex differentiable functions are known as *analytic*.
- We will be concerned with functions defined by well-behaved limiting processes.
  - Wherever the limit converges the functions will be analytic.
- Even if the limit doesn't converge, the function can sometimes still be defined using *analytic continuation*.

Zeros and poles

- Let $f(z)$ be analytic around $\alpha \in \mathbb{C}$. We can uniquely represent f around α by a *Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n.$$

- Let $\mathrm{Ord}_f(\alpha)$ denote the minimal n such that $c_n \neq 0$.
    - When $\mathrm{Ord}_f(\alpha) > 0$, α is known as a *zero*.
    - When $\mathrm{Ord}_f(\alpha) < 0$, α is known as a *pole*, and if $\mathrm{Ord}_f(\alpha) = -1$, α is known as a *simple pole*.
- $\mathrm{Res}_f(\alpha) \coloneqq c_{-1}$ is known as the *residue* of the pole at α.
- For simple poles, $\mathrm{Res}_f(\alpha) = \lim_{z \to \alpha} (z-\alpha) f(z)$.

M-L inequality

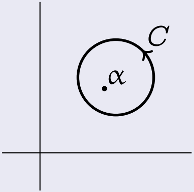
- The *maximum times length* (M-L) estimate is a simple way of bounding the absolute value of integrals:

$$\left|\int_C f(z)\,\mathrm{d}z\right| \leqslant M\,\mathrm{len}(C)$$

where $|f(z)| \leqslant M$ for z on C.

Cauchy's residue theorem

- Let C be a counterclockwise circle containing α, and let $f(z)$ be analytic on and within C, except for a pole at α.



- *Cauchy's residue theorem* states that

$$\int_C f(z)\,\mathrm{d}z = 2\pi i\,\mathrm{Res}_f(\alpha).$$

- In addition, C can be freely deformed in the analytic region.

Absolute value of complex exponentials

- To compute the norm of n^z, replace z with its real part:

$$\begin{aligned} |n^z| &= \left| e^{(x+iy)\ln n} \right| \\ &= e^{x \ln n} |\cos(y \ln n) + i \sin(y \ln n)| \\ &= n^x \sqrt{\cos^2(y \ln n) + \sin^2(y \ln n)} \\ &= n^x \end{aligned}$$

Four important functions

$$\zeta(z) \coloneqq \sum_{n=1}^{\infty} \frac{1}{n^z}$$

$$\Phi(z) \coloneqq \sum_{p} \frac{\ln p}{p^z}$$

$$\vartheta(x) \coloneqq \sum_{p \leqslant x} \ln p$$

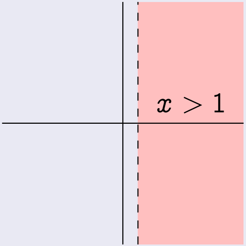
$$\pi(x) \coloneqq \sum_{p \leqslant x} 1$$

Convergence of $\zeta(z)$ and $\Phi(z)$

- $\zeta(z)$ converges absolutely for $x > 1$ by the integral test:

$$\sum_{n=1}^{\infty}\left|\frac{1}{n^z}\right| = \sum_{n=1}^{\infty}\frac{1}{n^x} \leqslant 1 + \int_1^{\infty}\frac{\mathrm{d}u}{u^x} = \frac{x}{x-1}.$$

  - Similarly, $\Phi(z)$ converges absolutely for $x > 1$.
- The region of convergence for $\zeta(z)$ and $\Phi(z)$:



Main steps of the proof

Setup

I $\zeta(z) = \prod_p (1 - 1/p^z)^{-1}$ for $x > 1$

II $\zeta(z)$ is analytic for $x \geqslant 1$, except for a simple pole at $z = 1$

III $\Phi(z)$ is analytic for $x \geqslant 1$, except for a simple pole at $z = 1$

IV $\vartheta(x) = O(x)$

Analytic Theorem

V If $|f(t)|$ is bounded and $\int_0^\infty f(t) e^{-zt}\, \mathrm{d}t$ can be analytically continued to $x = 0$ then $\int_0^\infty f(t)\, \mathrm{d}t$ converges

Downhill

VI $\int_1^\infty \frac{\vartheta(x) - x}{x^2}\, \mathrm{d}x$ converges

VII $\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1$

VIII $\lim_{x \to \infty} \frac{\pi(x)}{x / \ln x} = 1$

I. $\zeta(z) = \prod_p (1 - 1/p^z)^{-1}$ for $x > 1$

- Euler discovered a beautiful relation between $\zeta(z)$ and the primes:

$$\sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_p \frac{1}{1 - 1/p^z}$$

I. $\zeta(z) = \prod_p (1 - 1/p^z)^{-1}$ for $x > 1$

- Use the geometric series formula with ratio $1/p^z$:

$$\prod_p \frac{1}{1 - 1/p^z} = \prod_p \sum_{k \geqslant 0} \frac{1}{p^{kz}}$$

- Multiplying the first two factors:

$$\sum_{k \geqslant 0} \frac{1}{2^{kz}} \cdot \sum_{k \geqslant 0} \frac{1}{3^{kz}} = \sum_{k_1, k_2 \geqslant 0} \frac{1}{(2^{k_1} 3^{k_2})^z}$$

- Continue multiplying together the factors to obtain:

$$\sum_{k_1, k_2, k_3, \ldots \geqslant 0} \frac{1}{(2^{k_1} 3^{k_2} 5^{k_3} \cdots)^z}$$

- By rearranging terms and using unique factorization, this is equal to $\sum_{n \geqslant 1} 1/n^z$.

II. $\zeta(z)$ is analytic for $x \geqslant 1$, except for a simple pole at $z = 1$

- We want to show that $\zeta(z)$ admits an analytic continuation to the line $x = 1$.
  - Actually, this isn't possible because of a pole at $z = 1$.
- However, by "subtracting off" the pole, as in

$$\zeta(z) - \frac{1}{z-1},$$

  then the analytic continuation to $x = 1$ works.

II. $\zeta(z)$ is analytic for $x \geqslant 1$, except for a simple pole at $z = 1$

- For $x > 1$ we have

$$\zeta(z) - \frac{1}{z-1} = \sum_{n=1}^{\infty} \frac{1}{n^z} - \int_1^{\infty} \frac{\mathrm{d}u}{u^z} = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{n^z} - \frac{1}{u^z}\,\mathrm{d}u.$$

- The absolute value of this summand is at most

$$\begin{aligned} \max_{u\in[n,n+1]} \left| \frac{1}{n^z} - \frac{1}{u^z} \right| &= \max_{u\in[n,n+1]} \left| \int_n^u \frac{z\,\mathrm{d}v}{v^{z+1}} \right| \\ &\leqslant \max_{u\in[n,n+1]} \max_{v\in[n,u]} \left| \frac{z}{v^{z+1}} \right| \\ &= \frac{|z|}{n^{x+1}}. \end{aligned}$$

  - Employing the M-L inequality twice.
- By comparison with $\sum_{n\geqslant 1} \frac{|z|}{n^{x+1}}$, the sum on the right converges absolutely for $x > 0$.

III. $\Phi(z)$ is analytic for $x \geqslant 1$, except for a simple pole at $z = 1$

- Taking the logarithmic derivative of the Euler product:

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{\mathrm{d}}{\mathrm{d}z} \ln \zeta(z) = -\sum_p \frac{\ln p / p^z}{1 - 1/p^z} = -\Phi(z) - A(z)$$

where $A(z) := \sum_p \frac{\ln p}{p^z(p^z - 1)}$ is analytic for $x > 1/2$.

- The logarithmic derivative has an especially simple Laurent expansion around α:

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{\mathrm{Ord}_\zeta(\alpha)}{z - \alpha} + O(1)$$

- It follows $\mathrm{Res}_\Phi(\alpha) = -\mathrm{Ord}_\zeta(\alpha)$.
    - To show Φ is analytic at α, we show $\mathrm{Ord}_\zeta(\alpha) = 0$.

III. $\Phi(z)$ is analytic for $x \geqslant 1$, except for a simple pole at $z = 1$

- From

$$\lim_{\epsilon \to 0^+} \epsilon \sum_p \frac{\ln p}{p^{z+\epsilon}} = \mathrm{Res}_\Phi(z)$$

one gets that

$$\lim_{\epsilon \to 0^+} \epsilon \sum_p \frac{\ln p}{p^{x+\epsilon}} (p^{2yi} + 4p^{yi} + 6 + 4p^{-yi} + p^{-2yi})$$

$$= \mathrm{Res}_\Phi(x - 2yi) + 4\,\mathrm{Res}_\Phi(x - yi) + 6\,\mathrm{Res}_\Phi(x) + 4\,\mathrm{Res}_\Phi(x + yi) + \mathrm{Res}_\Phi(x + 2yi)$$

$$= 2\,\mathrm{Res}_\Phi(x + 2yi) + 8\,\mathrm{Res}_\Phi(x + yi) + 6\,\mathrm{Res}_\Phi(x)$$

  - The residues of Φ being invariant under complex conjugation being a consequence of $\Phi(z) = \overline{\Phi(\overline{z})}$.

- The ugly factor simplifies to $(p^{yi/2} + p^{-yi/2})^4$.
  - Nonnegative since its inside is real.

III. $\Phi(z)$ is analytic for $x \geqslant 1$, except for a simple pole at $z = 1$

- Then

$$\epsilon \sum_p \frac{\ln p}{p^{x+\epsilon}} (p^{yi/2} + p^{-yi/2})^4 \geqslant 0$$

and, taking the limit as $\epsilon \to 0^+$, we have

$$2\,\mathrm{Res}_\Phi(x + 2yi) + 8\,\mathrm{Res}_\Phi(x + yi) + 6\,\mathrm{Res}_\Phi(x) \geqslant 0$$

$$2\,\mathrm{Ord}_\zeta(x + 2iy) + 8\,\mathrm{Ord}_\zeta(x + iy) + 6\,\mathrm{Ord}_\zeta(x) \leqslant 0$$

  - Making use of $\mathrm{Res}_\Phi(z) = -\mathrm{Ord}_\zeta(z)$.

III. $\Phi(z)$ is analytic for $x \geqslant 1$, except for a simple pole at $z = 1$

- Using $\mathrm{Ord}_\zeta(z) \geqslant -1$, this becomes

$$2\,\mathrm{Ord}_\zeta(x + 2iy) + 8\,\mathrm{Ord}_\zeta(x + iy) \leqslant 6.$$

- For $x \geqslant 1$ and $z \neq 1$ we know $\zeta(z)$ has no poles, so

$$\mathrm{Ord}_\zeta(z) \geqslant 0.$$

- Combining the above two inequalities, we have

$$\mathrm{Ord}_\zeta(x + iy) = 0.$$

- Thus $\Phi(z)$ is analytic for $x \geqslant 1$ and $z \neq 1$.
  - $\Phi(1)$ is a simple pole with residue $-\mathrm{Ord}_\zeta(1) = 1$.

IV. $\vartheta(x) = O(x)$

- The binomial theorem gives the following upper bound on the central binomial coefficient:

$$\binom{2n}{n} \leqslant \sum_{k=0}^{2n} \binom{2n}{k} = (1+1)^{2n} = 4^n$$

- But every prime $n < p \leqslant 2n$ is a factor of $(2n)!/(n!)^2$:

$$\binom{2n}{n} \geqslant \prod_{n<p\leqslant 2n} p = \exp\left(\sum_{p\leqslant 2n} \ln p - \sum_{p\leqslant n} \ln p \right)$$
$$= \exp(\vartheta(2n) - \vartheta(n))$$

IV. $\vartheta(x) = O(x)$

- Combining these and taking the logarithm,

$$\vartheta(2n) - \vartheta(n) \leqslant n \ln 4.$$

- If n is a power of 2, summing this over n, $n/2$, $\ldots$, 1 the left side telescopes and the right side is a truncated geometric series:

$$\vartheta(2n) \leqslant (n + n/2 + \cdots + 1) \ln 4 \leqslant 2n \ln 4$$

- For arbitrary x, let $n \leqslant x < 2n$ where n is a power of 2. Then:

$$\vartheta(x) \leqslant \vartheta(2n) \leqslant 2n \ln 4 \leqslant 2x \ln 4$$

V. Analytic Theorem - The statement

- If $|f(t)|$ is bounded (say by M) and its Laplace transform

$$\int_0^\infty f(t)e^{-zt}\,\mathrm{d}t$$

defines an analytic function $g(z)$ for $x \geqslant 0$, then the integral converges for $z = 0$.

V. Analytic Theorem - Easy convergence

- Note the integral converges absolutely for $x > 0$ since

$$\int_0^\infty \left|f(t)e^{-zt}\right|\mathrm{d}t \leqslant M\int_0^\infty e^{-xt}\,\mathrm{d}t = \frac{M}{x}.$$

V. Analytic Theorem - Setup

- Define the 'truncated' Laplace transform,

$$g_T(z) := \int_0^T f(t)e^{-zt}\,\mathrm{d}t.$$

  This is analytic for all z by differentiation under the integral sign. As $T \to \infty$ this converges to $g(z)$ for $x > 0$.

- We will show this also occurs at $z = 0$, i.e.,

$$\lim_{T\to\infty} \int_0^T f(t)\,\mathrm{d}t = g(0).$$

V. Analytic Theorem - A useful function

- Consider the function

$$\big(g(z) - g_T(z)\big)\, e^{zT} \left(\frac{1}{z} + \frac{z}{R^2}\right).$$

- This is analytic for $x \geqslant 0$ and $z \neq 0$, since:
  - $g(z) - g_T(z)$ is analytic for $x \geqslant 0$
  - e^{zT} is analytic for all z
  - $1/z + z/R^2$ is analytic for $z \neq 0$
- At $z = 0$, this has a simple pole with residue $g(0) - g_T(0)$.

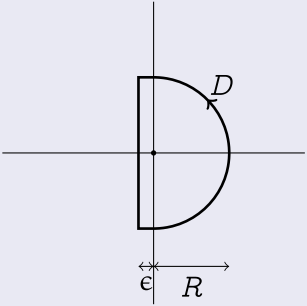
V. Analytic Theorem - A useful fact

- On the circle $|z| = R$, the final factor is equal to

$$\frac{x - yi}{x^2 + y^2} + \frac{x + yi}{x^2 + y^2} = \frac{2x}{R^2}.$$

V. Analytic Theorem - A useful contour

- Define the contour D in the complex plane by:



- Here ϵ (depending on R) is taken small enough so that $g(z)$ is analytic on and inside D.
- By Cauchy's residue theorem,

$$\int_D \big(g(z) - g_T(z)\big) e^{zT} \left(\frac{1}{z} + \frac{z}{R^2}\right) \mathrm{d}z = 2\pi i \big(g(0) - g_T(0)\big).$$

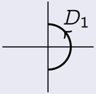
V. Analytic Theorem - A useful integral

- We split the integral

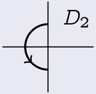
$$I := \int_D \left(g(z) - g_T(z)\right) e^{zT} \left(\frac{1}{z} + \frac{z}{R^2}\right) \mathrm{d}z$$

into three such that $I = I_1 - I_2 + I_3$:

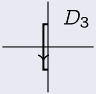
$$I_1 := \int_{D_1} \left(g(z) - g_T(z)\right) e^{zT} \left(\frac{1}{z} + \frac{z}{R^2}\right) \mathrm{d}z$$



$$I_2 := \int_{D_2} g_T(z) e^{zT} \left(\frac{1}{z} + \frac{z}{R^2}\right) \mathrm{d}z$$



$$I_3 := \int_{D_3} g(z) e^{zT} \left(\frac{1}{z} + \frac{z}{R^2}\right) \mathrm{d}z$$



V. Analytic Theorem - The first integral

- For $x > 0$, we have that $|g(z) - g_T(z)|$ is equal to

$$\left|\int_T^\infty f(t)e^{-zt}\,\mathrm{d}t\right| \leqslant M\int_T^\infty e^{-xt}\,\mathrm{d}t = M\frac{e^{-xT}}{x}.$$

- Using M-L we can bound $|I_1|$:

$$\left|\int_{D_1} \left(g(z) - g_T(z)\right) e^{zT}\left(\frac{1}{z} + \frac{z}{R^2}\right)\mathrm{d}z\right|$$
$$\leqslant \max_{z\in D_1} \pi R \quad M\frac{e^{-xT}}{x} \quad e^{xT} \quad \frac{2x}{R^2} \quad = O\left(\frac{1}{R}\right)$$

V. Analytic Theorem - The second integral

- For $x < 0$, we have that $|g_T(z)|$ is equal to

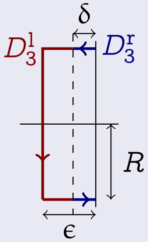
$$\left|\int_0^T f(t)e^{-zt}\,\mathrm{d}t\right| \leqslant M\int_0^T e^{-xt}\,\mathrm{d}t = M\frac{e^{-xT}-1}{|x|}.$$

- Using M-L we can bound $|I_2|$:

$$\left|\int_{D_2} g_T(z) \quad e^{zT}\left(\frac{1}{z}+\frac{z}{R^2}\right)\mathrm{d}z\right|$$
$$\leqslant \max_{z\in D_2} \pi R\; M\frac{e^{-xT}}{|x|}\; e^{xT} \quad \frac{2|x|}{R^2} \qquad = O\left(\frac{1}{R}\right)$$

V. Analytic Theorem - The third integral

- Since $g(z)\left(\frac{1}{z} + \frac{z}{R^2}\right)$ is analytic on D_3, its absolute value must be bounded (say by B).
- We have $|e^{zT}| \leqslant 1$ for z on D_3, but this is not sufficient for using M-L. Instead, we split D_3 into a left and right part based on a parameter $\delta \in (0, \epsilon)$:



V. Analytic Theorem - The third integral

- Using M-L on D_3^{l} and D_3^{r} separately we can bound $|I_3|$:

$$
\begin{array}{lcccccc}
 & \Big|\int_C & g(z) & \left(\frac{1}{z}+\frac{z}{R^2}\right) & e^{zT} & \mathrm{d}z\Big| & \\
\text{for } D_3^{\mathrm{l}}: & \leqslant 2(R+\epsilon) & & B & e^{-\delta T} & & = O(e^{-\delta T}) \\
\text{for } D_3^{\mathrm{r}}: & \leqslant \quad 2\delta & & B & 1 & & = O(\delta)
\end{array}
$$

- Taking the limit as $T \to \infty$,

$$0 \leqslant \limsup_{T\to\infty} |I_3| \leqslant O(\delta).$$

- Taking the limit as $\delta \to 0^+$,

$$\lim_{T\to\infty} |I_3| = 0.$$

V. Analytic Theorem - Conclusion

- We have

$$|g(0) - g_T(0)| = \left|\frac{I}{2\pi i}\right| \leqslant |I_1| + |I_2| + |I_3|.$$

- Taking the limit as $T \to \infty$,

$$0 \leqslant \limsup_{T\to\infty} |g(0) - g_T(0)| \leqslant O\left(\frac{1}{R}\right).$$

- Taking the limit as $R \to \infty$,

$$\lim_{T\to\infty} |g(0) - g_T(0)| = 0.$$

- Therefore

$$\int_0^\infty f(t)\,\mathrm{d}t = g(0).$$

VI. $\int_1^\infty \frac{\vartheta(x)-x}{x^2}\,\mathrm{d}x$ converges

- Note the Laplace transform of $\vartheta(e^t)$ is $\Phi(z)/z$:

$$\frac{\Phi(z)}{z} = \int_0^\infty \vartheta(e^t)e^{-zt}\,\mathrm{d}t$$

  - This follows via summation by parts, algebra, and the substitution $u = e^t$:

$$\begin{aligned}\Phi(z) &= \sum_{n=1}^{\infty} \vartheta(n)\left(\frac{1}{n^z} - \frac{1}{(n+1)^z}\right) + \lim_{n\to\infty}\frac{\vartheta(n)}{n^z} \\ &= \sum_{n=1}^{\infty} \vartheta(n)\int_n^{n+1} \frac{z\,\mathrm{d}u}{u^{z+1}} \qquad (x > 1) \\ &= z\int_1^\infty \frac{\vartheta(u)}{u^{z+1}}\,\mathrm{d}u\end{aligned}$$

VI. $\int_1^\infty \frac{\vartheta(x)-x}{x^2}\,\mathrm{d}x$ converges

- Substituting $z \mapsto z+1$ and subtracting $1/z = \int_0^\infty e^{-zt}\,\mathrm{d}t$, this becomes:

$$\frac{\Phi(z+1)}{z+1} - \frac{1}{z} = \int_0^\infty \left(\frac{\vartheta(e^t)}{e^t} - 1\right) e^{-zt}\,\mathrm{d}t$$

- By III, the left hand side is analytic for $x \geqslant 0$, since $\frac{\Phi(z+1)}{z+1}$ is analytic there.
  - Except at $z = 0$, where it has a simple pole with residue $\lim_{z\to 0} z\frac{\Phi(z+1)}{z+1} = 1$.
- By IV, the Laplace integrand is $O(e^t)/e^t - 1 = O(1)$.

VI. $\int_1^\infty \frac{\vartheta(x)-x}{x^2}\,\mathrm{d}x$ converges

- Using the analytic theorem with $f(t) := \vartheta(e^t)/e^t - 1$,

$$\int_0^\infty \frac{\vartheta(e^t)}{e^t} - 1\,\mathrm{d}t = \int_1^\infty \frac{\vartheta(x)-x}{x^2}\,\mathrm{d}x$$

  converges (the substitution $x = e^t$ was used).
  - This is a strong condition; if $\vartheta(x) = (1+\epsilon)x$ then this integral would be $\int_1^\infty \epsilon/x\,\mathrm{d}x$, which diverges.

VII. $\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1$

- Suppose otherwise. Then there is some $\epsilon > 0$ such that $\vartheta(x) > (1+\epsilon)x$ or $\vartheta(x) < (1-\epsilon)x$ for arbitrarily large x.
- In the first case, say $\{x_n\}$ is an increasing sequence with $\vartheta(x_n) > (1+\epsilon)x_n$. Since $\vartheta(x)$ is non-decreasing,

$$\int_{x_n}^{(1+\epsilon)x_n} \frac{\vartheta(x) - x}{x^2}\,\mathrm{d}x > \int_{x_n}^{(1+\epsilon)x_n} \frac{(1+\epsilon)x_n - x}{x^2}\,\mathrm{d}x$$
$$= \epsilon - \ln(1+\epsilon)$$

  which is a positive number not depending on n.
  - Since $\int_1^\infty \frac{\vartheta(x)-x}{x^2}\,\mathrm{d}x$ converges, the above sequence of subintegrals should go to zero as $n \to \infty$.

VII. $\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1$

- In the second case, say $\{x_n\}$ is an increasing sequence with $\vartheta(x_n) < (1-\epsilon)x_n$. Since $\vartheta(x)$ is non-decreasing,

$$\int_{(1-\epsilon)x_n}^{x_n} \frac{\vartheta(x)-x}{x^2}\,\mathrm{d}x < \int_{(1-\epsilon)x_n}^{x_n} \frac{(1-\epsilon)x_n - x}{x^2}\,\mathrm{d}x$$
$$= \epsilon + \ln(1-\epsilon)$$

  which is a negative number not depending on n.
  - Since $\int_1^\infty \frac{\vartheta(x)-x}{x^2}\,\mathrm{d}x$ converges, the above sequence of subintegrals should go to zero as $n \to \infty$.
- Either case leads to a contradiction, so we must in fact have $\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1$.

VIII. $\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1$

- By multiplying the number of nonzero terms with the maximum or minimum term one derives upper and lower bounds on $\vartheta(x)$ in terms of $\pi(x)\ln x$:

$$\vartheta(x) = \sum_{p\leqslant x} \ln p \leqslant \pi(x)\ln x$$

$$\vartheta(x) \geqslant \sum_{x^{1-\epsilon}<p\leqslant x} \ln p \geqslant \left(\pi(x) - \pi(x^{1-\epsilon})\right)\ln(x^{1-\epsilon})$$

  - For the lower bound, we ignore primes less than $x^{1-\epsilon}$ for some constant parameter $\epsilon \in (0,1)$.

VIII. $\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1$

- Combining these,

$$\frac{\vartheta(x)}{x} \leqslant \frac{\pi(x)}{x/\ln x} \leqslant \frac{1}{1-\epsilon}\frac{\vartheta(x)}{x} + \frac{\ln x}{x^{\epsilon}}.$$

- Taking the limit as $x \to \infty$ and using $\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1$,

$$1 \leqslant \limsup_{x\to\infty} \frac{\pi(x)}{x/\ln x} \leqslant \frac{1}{1-\epsilon}.$$

  - Similarly for the lim inf.
- Taking the limit as $\epsilon \to 0^{+}$,

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1.$$