The projective plane that wasn't there: How Lam's problem was solved

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Abstract

A *projective plane* is a collection of points and lines such that (a) every pair of lines meets in a unique point, and (b) there is a unique line through every pair of points. If the collection is finite, the plane is said to be of *order* n when every line passes through exactly $n+1$ points. It is well known that projective planes exist in every prime power order and do not exist in order 6, making $n = 10$ the first uncertain case. *Lam's problem* is to determine whether a projective plane of order 10 exists. The problem was open for decades and finally solved through a spectacular combination of mathematical ingenuity and computational power (Lam, *American Mathematical Monthly*, 1991).

In this talk, we give a high-level overview of how Lam's remarkable nonexistence result was achieved. The solution begins with a hypothetical projective plane of order ten, uses it to generate a binary error-correcting code (a certain linear subspace of $\mathbb{F}_2^{111}$), carefully counts the number of codewords of each weight in this code, and finally runs computer searches starting from codewords of small weights. Extensive searches performed in the 1970s and 1980s came back empty, together implying that there are no projective planes of order ten. Although these searches relied on the correctness of computer code that cannot be considered completely trustworthy, computer-checkable certificates were produced in 2020 demonstrating the nonexistence of the projective plane of order ten (Bright et al., AAAI 2021).