Symbolic Sets for Proving Bounds on Rado Numbers

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Rado numbers

For a linear equation $\mathcal{E}$, the 3-colour *Rado number* $R_3(\mathcal{E})$ is the smallest integer n, if it exists, such that every 3-colouring of the integers $[1, n] = \{1, \ldots, n\}$ contains a monochromatic solution to $\mathcal{E}$.

For $\mathcal{E}\colon x + 3y = 3z$, we have $R_3(\mathcal{E}) = 27$, since

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provides a 3-colouring of $[1, 26]$, but there is no way to colour $[1, 27]$ without introducing a monochromatic solution to $\mathcal{E}$.

Results

We focused on the equations $ax + ay = bz$ and $ax + by = bz$.

We have computed a number of previously unknown exact values using SAT solvers, extending the results presented by Chang, De Loera, and Wesley (ISSAC 2022) to the values of

- $R_3(ax + ay = bz)$ for $1 \leq b \leq 15$ and $1 \leq a \leq 30$, and
- $R_3(ax + by = bz)$ for $1 \leq a \leq 15$ and $1 \leq b \leq 25$.

Results

Guided by the satisfying assignments generated by the SAT solver in the process of computing these values, we proved the following two theorems.

- For coprime positive integers a, b with $a > b \geq 3$ and $a^2 + a + b > b^2 + ba$,

$$R_3(ax + by = bz) \geq a^3 + a^2 + (2b + 1)a + 1.$$

- For odd integers $a \geq 7$,

$$R_3(ax + ay = (a + 1)z) \geq a^3(a + 1).$$

The Rado Value Problem in SAT

The formula: Given positive integers n and a linear equation $\mathcal{E}$, we construct a formula in CNF that is satisfiable if and only if there exists a 3-colouring of $[1, n]$ avoiding monochromatic solutions of $\mathcal{E}$. Therefore,

if the formula for $[1, n-1]$ is satisfiable then $R_3(\mathcal{E}) \geq n$; and

if the formula for $[1, n]$ is unsatisfiable then $R_3(\mathcal{E}) \leq n$.

Variables: Variables are denoted by $v_{0,j}$, $v_{1,j}$, and $v_{2,j}$ with $1 \leq j \leq n$ and the meaning that $v_{i,j}$ is true if and only if colour i is assigned to integer j. There are $3n$ variables in the formula.

SAT Clauses for Rado

- **At least one colour is assigned to every integer:** For each integer $j \in [1, n]$, the clause

$$v_{0,j} \vee v_{1,j} \vee v_{2,j}$$

  ensures at least one colour is assigned to j.

- **At most one colour is assigned to every integer:** For each integer $j \in [1, n]$, the clauses

$$\bigwedge_{0 \leq i_1 < i_2 < 3} (\neg v_{i_1,j} \vee \neg v_{i_2,j})$$

  ensure at most one colour $i \in \{0, 1, 2\}$ is assigned to j.

- **There is no monochromatic solution to $\mathcal{E}$:** For each colour $i \in \{0, 1, 2\}$ and for each solution (x, y, z) to $\mathcal{E}$, the clause

$$\neg v_{i,x} \vee \neg v_{i,y} \vee \neg v_{i,z}$$

  ensures the solution (x, y, z) is not monochromatic in colour i.

New Values for $R_3(ax + by = bz)$

$a\backslash b$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	14	14	27	64	125	216	343	512	729	1000	1331	1728	2197	2744	3375
2	43	14	31	14	125	27	343	64	729	125	1331	216	2197	343	3375
3	94	61	14	73	125	14	343	512	27	1000	1331	64	2197	2744	125
4	173	43	109	14	141	31	343	14	729	125	1331	27	2197	343	3375
5	286	181	186	180	14	241	343	512	729	14	1331	1728	2197	2744	27
6	439	94	43	61	300	14	379	73	31	125	1331	14	2197	343	125
7	638	428	442	456	470	462	14	561	729	1000	1331	1728	2197	14	3375
8	889	173	633	43	665	109	644	14	793	141	1331	31	2197	343	3375
9	1198	856	94	892	910	61	896	896	14	1081	1331	73	2197	2744	125
10	1571	286	1171	181	43	186	1190	180	1206	14	1431	241	2197	343	31
11	2014	1508	1530	1552	1574	1596	1618	1584	1575	1580	14	1849	2197	2744	3375
12	2533	439	173	94	2005	43	2053	61	109	300	2024	14	2341	379	141
13	3134	2432	2458	2484	2510	2536	2562	2588	2574	2530	2541	2544	14	2913	3375
14	3823	638	3039	428	3095	442	43	456	3207	470	3113	462	3146	14	3571
15	4606	3676	286	3736	94	181	3826	3856	186	61	3795	180	3835	3836	14
16	5489	889	**4465**	173	**4529**	633	**4593**	43	**4657**	665	**4576**	109	**4602**	644	**4620**
17	6478	**5288**	**5322**	**5356**	**5390**	**5424**	**5458**	**5492**	**5526**	**5560**	**5594**	**5424**	**5447**	**5474**	**5475**
18	7579	1198	439	856	**6355**	94	**6427**	892	43	910	**6571**	61	**6409**	896	300
19	8798	**7316**	**7354**	**7392**	**7430**	**7468**	**7506**	**7544**	**7582**	**7620**	**7658**	**7696**	**7488**	**7518**	**7530**
20	10141	1571	**8541**	286	173	1171	**8701**	181	**8781**	43	**8861**	186	**8941**	1190	109
21	11614	**9808**	638	**9892**	**9934**	428	94	**10060**	442	**10144**	**10186**	456	**10270**	61	470
22	13223	2014	**11287**	1508	**11375**	1530	**11463**	1552	**11551**	1574	43	1596	**11727**	1618	**11535**
23	14974	**12812**	**12858**	**12904**	**12950**	**12996**	**13042**	**13088**	**13134**	**13180**	**13226**	**13272**	**13318**	**13364**	**13110**
24	16873	2533	889	439	**14665**	173	**14761**	94	633	2005	**14953**	43	**15049**	2053	665
25	18926	**16376**	**16426**	**16476**	286	**16576**	**16626**	**16676**	**16726**	181	**16826**	**16876**	**16926**	**16976**	186
26	21139	3134	**18435**	2432	**18539**	2458	**18643**	2484	**18747**	2510	**18851**	2536	43	2562	**19059**
27	23518	**20548**	1198	**20656**	**20710**	856	**20818**	**20872**	94	**20980**	**21034**	892	**21142**	**21196**	910
28	26069	3823	**22933**	638	**23045**	3039	173	428	**23269**	3095	**23381**	442	**23493**	43	**23605**
29	28798	**25376**	**25434**	**25492**	**25550**	**25608**	**25666**	**25724**	**25782**	**25840**	**25898**	**25956**	**26014**	**26072**	**26130**
30	31711	4606	1571	3676	439	286	**28351**	3736	1171	94	**28591**	181	**28711**	3826	43

Table 1
$R_3(\mathcal{E}(3, 0; a, b, b))$ for $1 \leq b \leq 15$ and $1 \leq a \leq 30$. The previously unknown values are presented in boldface, and the underlined entries correspond to equations whose coefficients are not coprime.

New Values for $R_3(ax + ay = bz)$

$b \backslash a$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	14	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
2	1	14	243	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
3	54	54	14	384	2000	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
4	∞	1	108	14	875	243	4459	∞	∞	∞	∞	∞	∞	∞	∞
5	∞	105	135	180	14	864	3430	3072	12393	∞	∞	∞	∞	∞	∞
6	∞	54	1	54	750	14	3087	384	243	2000	**27951**	∞	∞	∞	∞
7	∞	455	336	308	875	756	14	1536	8748	7500	**23958**	**10368**	**54925**	∞	∞
8	∞	∞	432	1	1000	108	2744	14	8019	875	**21296**	243	**48334**	4459	**97875**
9	∞	∞	54	585	1125	54	3087	1224	14	6000	**18634**	384	**41743**	**30184**	2000
10	∞	∞	1125	105	1	135	3430	180	7290	14	**17303**	864	**37349**	3430	243
11	∞	∞	2019	847	1958	1188	**3773**	**1672**	**8019**	**5500**	14	**6048**	**35152**	**24696**	**77625**
12	∞	∞	∞	54	2400	1	**4116**	54	108	750	**15972**	14	**32955**	3087	875
13	∞	∞	∞	1710	3445	1963	**4459**	**1456**	**9477**	**6500**	**17303**	**5616**	14	**21952**	**60750**
14	∞	∞	∞	455	3675	336	1	308	**10206**	875	**18634**	756	**30758**	14	**57375**
15	∞	∞	∞	5408	54	105	**6615**	**2760**	135	54	**19965**	180	**32955**	**20580**	14
16	∞	∞	∞	∞	5725	432	**7616**	1	**11664**	1000	**21296**	108	**35152**	2744	**54000**
17	∞	∞	∞	∞	8330	4743	**10064**	**3825**	**12393**	**8500**	**22627**	**7344**	**37349**	**23324**	**57375**
18	∞	∞	∞	∞	12069	54	**10962**	585	1	1125	**23958**	54	**39546**	3087	750
19	∞	∞	∞	∞	16397	6726	**14782**	**4332**	**16853**	**9500**	**25289**	**8208**	**41743**	**26068**	**60750**
20	∞	∞	∞	∞	∞	1125	**14700**	105	**19080**	1	**26620**	135	**43940**	3430	108
21	∞	∞	∞	∞	∞	455	54	**6699**	336	**12579**	**27951**	308	**46137**	54	875
22	∞	∞	∞	∞	∞	**2019**	**20580**	**847**	**25047**	**1958**	1	**1188**	**48334**	**3773**	**74250**
23	∞	∞	∞	∞	∞	**19056**	**27853**	**8556**	**32453**	**17250**	**35926**	**9936**	**50531**	**31556**	**77625**
24	∞	∞	∞	∞	∞	∞	**28956**	54	432	**2400**	**39600**	1	**52728**	**4116**	1000
25	∞	∞	∞	∞	∞	∞	**40163**	**10200**	**42975**	105	**47850**	**12850**	**54925**	**34300**	135

Table 2

$R_3(\mathcal{E}(3,0;a,a,b))$ for $1 \le a \le 15$ and $1 \le b \le 25$. The previously unknown values are presented in boldface, and the underlined entries correspond to equations whose coefficients are not coprime.

SAT Solvers and Resources

- Used CaDiCaL (Biere 2018) (via PySAT (SAT 2018)) for incremental SAT solving and Kissat (Biere 2021) for non-incremental runs.
- Approach: For each n, incrementally checked satisfiability until the first unsatisfiable instance, which determines the Rado number.
- The largest instance solved (showing $R_3(15x + 15y = 8z) = 97875$) contained 293,625 variables and 255,884,401 clauses, requiring 58.8 hours to solve.
- Total SAT instance generation time: $\sim$11 hours.
- Total CPU time for unsatisfiable instances: $\sim$251 hours; for satisfiable instances: $\sim$53 hours.

Patterns in $R_3(ax + by = bz)$

Analysis of computed $R_3(ax + by = bz)$ reveals explicit formulas for each b.

Observation: For coprime a and b, the Rado number often takes the form $a^3 + a^2 + (2b + 1)a + 1$.

- $b = 2, \quad a \in [7, 30]: \quad a^3 + a^2 + 5a + 1$
- $b = 3, \quad a \in [4, 30]: \quad a^3 + a^2 + 7a + 1$
- $b = 4, \quad a \in [7, 30]: \quad a^3 + a^2 + 9a + 1$
- $b = 5, \quad a \in [7, 30]: \quad a^3 + a^2 + 11a + 1$
- $b = 6, \quad a \in [11, 30]: \quad a^3 + a^2 + 13a + 1$
- $b = 7, \quad a \in [11, 30]: \quad a^3 + a^2 + 15a + 1$
- $b = 8, \quad a \in [13, 30]: \quad a^3 + a^2 + 17a + 1$

Pattern Discovery and Conjecture

- The table patterns were uncovered via curve fitting and analyzing differences in the sequences.
- Constant differences suggested linearity, while constant third differences indicated cubic dependence.
- **Conjecture:** For coprime a, b with $a^2 + a + b > b^2 + ba$ and $a > b \geq 3$, and $\mathcal{E}$ representing $ax + by = bz$,

$$R_3(\mathcal{E}) = a^3 + a^2 + (2b + 1)a + 1.$$

We prove the **lower bound** in this conjecture (replace $=$ by $\geq$).

Automating Case-Based Analysis

The proofs of our theorems involve a large number of tedious proof by cases. Shallit (2021) proposed that such case-by-case analyses are better performed by automated search algorithms to enhance both efficiency and correctness.

In this direction, we employ an automated verification approach using our tool AutoCase to verify case-based proofs. To check the proofs the tool needs support for operations on *symbolic sets* of the form $[1, n]$ where n is a *symbolic* integer.

Note that typical computer algebra systems like Maple or Mathematica seem to lack the ability to even express the set $[1, n]$ when n is symbolic, let alone do operations like union and intersection on such sets.

Proof Strategy

Theorem: For coprime a, b with $a^2 + a + b > b^2 + ba$ and $a > b \geq 3$, and $\mathcal{E}$ representing $ax + by = bz$,

$$R_3(\mathcal{E}) \geq a^3 + a^2 + (2b + 1)a + 1.$$

Proof Strategy: Find a colouring of the set

$$[1, a^3 + a^2 + (2b + 1)a]$$

that has no monochromatic solutions of equation $\mathcal{E}$.

Issue: Note that a and b here are *symbolic*, so the set $[1, a^3 + a^2 + (2b + 1)a]$ is a symbolic range of integers. The colouring of the integers in this set must also be symbolic.

An Example Bound

As an example, consider $\mathcal{E} : ax + y = z$ and the bound

$$R_3(\mathcal{E}) \geq a^3 + 5a^2 + 7a + 1.$$

Divide the set $[1, a^3 + 5a^2 + 7a]$ into the following seven intervals.

$$\begin{aligned}
&P_0 = [1, a], \quad P_1 = [a + 1, a^2 + 2a], \\
&P_2 = [a^2 + 2a + 1, a^2 + 3a], \\
&P_3 = [a^2 + 3a + 1, a^3 + 4a^2 + 4a], \\
&P_4 = [a^3 + 4a^2 + 4a + 1, a^3 + 4a^2 + 5a], \\
&P_5 = [a^3 + 4a^2 + 5a + 1, a^3 + 5a^2 + 6a], \\
&P_6 = [a^3 + 5a^2 + 6a + 1, a^3 + 5a^2 + 7a].
\end{aligned}$$

Colour the integers in $P_0 \cup P_2 \cup P_4 \cup P_6$ red, the integers in $P_1 \cup P_5$ blue, and the integers in P_3 green.

An Example Colouring

In order to verify that this colouring provides a valid lower bound, we need to verify:

- All integers in $[1, a^3 + 5a^2 + 7a]$ are coloured red (R), blue (B), or green (G). No integer was coloured two different colours.
- There are no all-red, all-blue, or all-green solutions of the equation $\mathcal{E} : ax + y = z$.

That is, $R \cup G \cup B = [1, a^3 + 5a^2 + 7a]$, R, B and G are pairwise disjoint, and $\{(x, y, z) : ax + y = z\}$ is disjoint with R^3, B^3, and G^3.

Proving with AutoCase

AutoCase takes as input the symbolic sets R, G, and B and verifies that the colouring meets all the required properties.

For example, in our last example, `SymPy` would verify that

$$|R| + |G| + |B| = a^3 + 5a^2 + 7a.$$

To check that there are no all-red solutions to $\mathcal{E}$, a `Z3` instance is created that says x, y, z are integers with $ax + y = z$, and $x, y, z \in R$. `Z3` verifies that the instance is unsatisfiable.

Conclusion

The colourings that the SAT solvers find can be used to provide lower bounds of Rado numbers.

In some cases, we were able to generalize these lower bounds to theorems parameterized by a symbolic integer a.

Computer algebra systems we tried did not have sufficient support for *symbolic* sets (subsets of $[1, a]$ where a is *symbolic*) so we developed a new SC^2 tool with support for the operations that we needed in order to automatically verify our general lower bounds.